Quantizing Fields

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Abstract

Quantum field theories emerged from the confluence of quantum mechanics and special relativity, and provide an amazingly accurate theoretical framework for describing the behaviour of subatomic particles and forces. This course will give an introduction into quantum field theory, both conceptually and technically. Canonical and covariant quantization methods will be discussed, with an emphasis on the path integral formulation, which finds manifold applications in both particle physics and condensed matter systems. Topics covered include the quantum-mechanical path integral, the quantization of bosonic and fermionic fields, functional techniques involving generating functionals and correlators, and perturbation theory in terms of Feynman diagrams.

These are notes to the course given at Utrecht University in 2012-2013.

Last update 16.01.13

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Chapter 1

Introduction

> It is no longer possible to present perturbative Field Theory pedagogically in one year – the days of the standard one-year course based on QED are gone forever.
>
> *Pierre Ramond*
> *Field Theory: a modern primer.*

The main recommended book

M. E. Peskin and D. V. Schroeder. "An Introduction to Quantum Field Theory", Perseus Books, The Advanced Book Program (Reading, MA).

Recommended Literature

[1] J. Cardy, *Scaling and Renormalization in Statistical Physics*, Cambridge University Press.

[2] D. Amit, *Field Theory, the Renormalization Group and Critical Phenomena*, World Scientific.

[3] L.Ryder, *Quantum Field Theory*, Cambridge University Press.

[4] M. Stone, *The Physics of Quantum Fields*, Graduate Texts in Contemporary Physics, Springer-Verlag.

[5] C. Itzykson and J. B. Zuber, *Quantum Field Theory*, McGraw-Hill.

[6] J. D. Bjorken and S. Drell, *Relativistic Quantum Fields*, McGraw-Hill.

[7] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Pergamon Press.

[8] R. P. Feynman, *Path Integrals and Quantum Mechanics*, McGraw Hill.

[9] G. Parisi, *Statistical Field Theory*, Addison Wesley.

[10] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Oxford University Press.

[11] S. Coleman, *Aspects of Symmetry*, Cambridge University Press.

[12] R. Rajaraman, *Solitons and Instantons*, North-Holland.

[13] A. M. Polyakov, *Gauge Fields and Strings*, Harwood.

[14] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory*, Springer-Verlag.

[15] R. Balian and J. Zinn-Justin, *Methods of Field Theory*, North-Holland.

[16] L. Schulman, *Techniques and Applications of Path Integration*, Wiley.

[17] C. Itzykson and J. Drouffe, *Statistical Field Theory*, Cambridge University Press.

[18] P. Ramond, *Field Theory: a Modern Primer*, Addison Wesley.

[19] S. J. Chang, *Introduction to Quantum Field Theory*, World Scientific.

[20] S. Doniach and E. H. Sondheimer, *Green's's Functions for Solid State Physicists*, Imperial College Press/ World Scientific.

[21] A. Abrikosov, L. Gorkov and I. Dzyaloshinsky, *Methods of Quantum Field Theory in Statistical Physics*, Dover.

[22] A. Fetter and J. D. Walecka, *Quantum Theory of Many Particle Systems*, McGraw-Hill.

[23] R. P. Feynman, *Statistical Mechanics*, Addison-Wesley.

[24] E. Fradkin, *Field Theories of Condensed Matter Systems*, Addison-Wesley.

[25] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics*, Addison Wesley.

[26] D. Pines and P. Nozieres, *The Theory of Quantum Liquids*, Addison Wesley-Perseus.

[27] J. Negele and H. Orland, *Quantum Many Particle Systems*, Addison Wesley.

[28] N. Goldenfeld, *Lectures on Phase Transitions ad the Renormalization Group*, Addison Wesley.

[29] C. Nash and S. Sen, *Topology and Geometry for Physicists*, Academic Press.

[30] S. Weinberg, *The Quantum Theory of Fields* (three volumes), Cambridge University Press.

[31] A. Zee, *Quantum Field Theory, in a nutshell*, Cambridge University Press.

[32] M. Kaku, *Quantum Field Theory*, McGraw-Hill.

Organization

The present course is organized as follows. The lectures are scheduled on Thursday 13.15-15.00. The classes are scheduled on Thusday 13.15-15.00 and on Tuesday 15.15-17.00 and 17.15-19.00. The lectures and classes will be in **BBL161**.

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Chapter 2

Classical fields and symmetries

This chapter is dedicated to reviewing of some of the important prerequisites for study quantum field theory. It includes the Lagrangian description of classical systems with an infinite number of degrees of freedom and the first Noether theorem, which allows one to construct dynamical invariants, *i.e.* the quantities remaining invariant under the time evolution. This material is assumed to be largely known and, therefore, is delegated for self-activating.

2.1 Continuous systems in classical mechanics

To describe continuous systems, such as vibrating solid, a transition to an infinite number of degrees of freedom is necessary. Indeed, one has to specify coordinates of all the points which are infinite in number. In fact, the continuum case can be reached by taking an appropriate limit of a system with a finite number of discrete coordinates. Our first example is an elastic rod of fixed length ℓ which undergoes small longitudinal vibrations. We approximate the rod by a system of equal mass m particles spaced a distance Δa apart and connected by uniform massless springs having the force constant k. The total length of the system is $\ell = (n+1)\Delta a$. We describe the displacement of the ith particle from its equilibrium position by the coordinate ϕ_i. Then the kinetic energy of the particles is

$$T = \sum_{i=1}^{n} \frac{m}{2} \dot{\phi}_i^2 \,.$$

The potential energy is stored into springs and it is given by the sum

$$U = \frac{1}{2} k \sum_{i=0}^{n} (\phi_{i+1} - \phi_i)^2 \,.$$

Here we associate $\phi_0 = 0 = \phi_{n+1}$ with the end points of the interval which do not move. The force acting on ith particle is $F_i = -\frac{\partial U}{\partial \phi_i}$:

$$F_i = k(\phi_{i+1} + \phi_{i-1} - 2\phi_i) \,.$$

This formula shows that the force exerted by the spring on the right of the ith particle equals to $k(\phi_{i+1} - \phi_i)$, while the force exerted from the left is $k(\phi_i - \phi_{i-1})$. The Lagrangian is

$$L = T - U = \sum_{i=1}^{n} \frac{m}{2} \dot{\phi}_i^2 - \frac{1}{2} k \sum_{i=0}^{n} (\phi_{i+1} - \phi_i)^2 \,.$$

At this stage we can take a continuum limit by sending $n \to \infty$ and $\Delta a \to 0$ so that $\ell = (n+1)\Delta a$ is kept fixed. Increasing the number of particles we will be increasing the total mass of a system. To keep the total mass finite, we assume that the ratio $m/\Delta a \to \mu$, where μ is a finite mass density. To keep the force between the particles finite, we assume that in the large particle limit $k\Delta a \to Y$, where Y is a finite quantity. Thus, we have

$$L = T - U = \frac{1}{2}\sum_{i=1}^{n} \Delta a \Big(\frac{m}{\Delta a}\Big)\dot{\phi}_i^2 - \frac{1}{2}\sum_{i=0}^{n} \Delta a (k\,\Delta a)\Big(\frac{\phi_{i+1}-\phi_i}{\Delta a}\Big)^2 .$$

Taking the limit, we replace the discrete index i by a continuum variable x. As a result, $\phi_i \to \phi(x)$. Also

$$\frac{\phi_{i+1}-\phi_i}{\Delta a} \to \frac{\phi(x+\Delta a)-\phi(x)}{\Delta a} \to \partial_x \phi(x)\,.$$

Thus, taking the limit we find

$$L = \frac{1}{2}\int_0^\ell \mathrm{d}x \left[\mu\dot{\phi}^2 - Y(\partial_x\phi)^2\right].$$

Also equations of motion can be obtained by the limiting procedure. Starting from

$$\frac{m}{\Delta a}\ddot{\phi}_i - k\Delta a \frac{\phi_{i+1}+\phi_{i-1}-2\phi_i}{\Delta a^2} = 0,$$

and using

$$\lim_{\Delta a\to 0} \frac{\phi_{i+1}+\phi_{i-1}-2\phi_i}{\Delta a^2} = \frac{\partial^2\phi}{\partial x^2} \equiv \partial_{xx}\phi$$

we obtain the equation of motion

$$\mu\ddot{\phi} - Y\,\partial_{xx}\phi = 0\,.$$

Just as there is a generalized coordinate ϕ_i for each i, there is a generalized coordinate $\phi(x)$ for each x. Thus, the finite number of coordinates ϕ_i has been replaced by a function of x. Since ϕ depends also on time, we are dealing with the function of two variables $\phi(x,t)$ which is called the *displacement field*. The Lagrangian is an integral over x of the *Lagrangian density*

$$\mathscr{L} = \frac{1}{2}\mu\dot{\phi}^2 - \frac{1}{2}Y(\partial_x\phi)^2\,.$$

The action is a functional of $\phi(x,t)$:

$$S[\phi] = \int_{t_1}^{t_2}\mathrm{d}t\int_0^\ell \mathrm{d}x\,\mathscr{L}(\phi(x,t),\dot{\phi}(x,t),\partial_x\phi(x,t))\,.$$

It is possible to obtain the equations of motion for the field $\phi(x,t)$ directly from the continuum Lagrangian. One has to understand how the action changes under an infinitesimal change of the field

$$\phi(x,t) \to \phi(x,t) + \delta\phi(x,t)\,. \tag{2.1}$$

The derivatives change accordingly,

$$\frac{\partial}{\partial t}\phi(x,t) \to \frac{\partial}{\partial t}\phi(x,t) + \frac{\partial}{\partial t}\delta\phi(x,t)\,, \tag{2.2}$$

$$\frac{\partial}{\partial x}\phi(x,t) \to \frac{\partial}{\partial x}\phi(x,t) + \frac{\partial}{\partial x}\delta\phi(x,t)\,. \tag{2.3}$$

This gives

$$\delta S[\phi] = S[\phi+\delta\phi] - S[\phi] = \int_{t_1}^{t_2} \mathrm{d}t \int_0^\ell \mathrm{d}x \Big[\frac{\partial \mathscr{L}}{\partial \phi}\delta\phi + \frac{\partial \mathscr{L}}{\partial \dot\phi}\partial_t\delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_x\phi)}\partial_x\delta\phi \Big] .$$

Integrating by parts, we find

$$\begin{aligned} \delta S[\phi] &= \int_{t_1}^{t_2} \mathrm{d}t \int_0^\ell \mathrm{d}x \Big[\frac{\partial \mathscr{L}}{\partial \phi} - \partial_t \frac{\partial \mathscr{L}}{\partial \dot\phi} - \partial_x \frac{\partial \mathscr{L}}{\partial(\partial_x\phi)} \Big]\delta\phi \\ &+ \int_0^\ell \mathrm{d}x \frac{\partial \mathscr{L}}{\partial(\partial_t\phi)}\delta\phi\big|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} \mathrm{d}t \frac{\partial \mathscr{L}}{\partial(\partial_x\phi)}\delta\phi\big|_{x=0}^{x=\ell} . \end{aligned} \tag{2.4}$$

The action principle requires that the action principle be stationary with respect to infinitezimal variations of the fields that leave the field values at the initial and finite time unaffected, *i.e.*

$$\delta\phi(x,t_1) = \delta\phi(x,t_2) = 0 .$$

On the other hand, since the rod is clamped, the displacement at the end points must be zero, *i.e.*

$$\delta\phi(0,t) = \delta\phi(\ell,t) = 0 .$$

Under these circumstances we derive the Euler-Lagrange equations for our continuum system

$$\frac{\partial}{\partial t}\Big(\frac{\partial \mathscr{L}}{\partial(\partial_t\phi)}\Big) + \frac{\partial}{\partial x}\Big(\frac{\partial \mathscr{L}}{\partial(\partial_x\phi)}\Big) - \frac{\partial \mathscr{L}}{\partial \phi} = 0 .$$

Let us now discuss the solution of the field equation

$$\ddot\phi - c^2\partial_{xx}\phi = 0 , \qquad c = \sqrt{\frac{Y}{\mu}} ,$$

where c is the propagation velocity of vibrations through the rod. This equation is linear and, for this reason, its solutions satisfy the superposition principle. Take an ansatz

$$\phi(x,t) = e^{ikx}a_k(t) + e^{-ikx}b_k(t) .$$

If we impose $\phi(0,t) = 0$, then $b_k(t) = -a_k(t)$ and we can refine the ansatz as

$$\phi(x,t) = a_k(t)\sin kx .$$

Requiring that $\phi(\ell,t) = 0$ we get $\sin k\ell = 0$, i.e. $k \equiv k_n = \frac{\pi n}{\ell}$. Coefficients $a_k(t)$ then obey

$$\ddot a_k + c^2k^2a_k(t) = 0 \qquad \to \qquad a_k(t) = e^{i\omega_k t}a_k ,$$

where $\omega_k = \pm ck$ is the dispersion relation. Thus, the general solution is

$$\phi(x,t) = \sum_n \sin k_n x\Big(A_n\cos\omega_n t + B_n\sin\omega_n t\Big) , \qquad \omega_n = ck_n ,$$

and the constants A_n, B_n are fixed by the initial conditions, which is an initial profile $\phi(x,0)$ and an initial velocity $\dot\phi(x,0)$.

Scalar and vector fields

The generalization to continuous systems in more space dimensions is now straightforward. In two-dimensions one can start with two-dimensional lattice of springs. The displacement of a particle at the site (i,j) is measured by the quantity $\vec{\phi}_{ij}$, which is a two-dimensional vector. In the limit when we go to a continuum, this becomes a displacement field $\vec{\phi}(x,y,t)$ of a membrane subjected to small vibrations in the (x,y)-plane. In three dimensions we get a vector $\vec{\phi}_{ijk}$. The continuous limit yields a three-dimensional displacement field $\vec{\phi}(x,y,z,t)$ of a continuous solid vibrating in the x,y,z directions with eoms of a partial differential equation type:

$$\ddot{\vec{\phi}} - c_1\partial_{xx}\vec{\phi} - c_2\partial_{yy}\vec{\phi} - c_3\partial_{zz}\vec{\phi} - c_4\partial_{xy}\vec{\phi} - c_5\partial_{yz}\vec{\phi} - c_6\partial_{xz}\vec{\phi} = 0\,,$$

the coefficients c_i encode the properties of the solid.

Tensors

In general, fields depending on the space-time variables are tensors, *i.e.* they transforms under general coordinate transformations in a definite way. Namely, a tensor field $\phi^{i_1\ldots i_p}_{j_1\ldots j_q}$ of rank (p,q) under general coordinate transformations of the coordinates x^i: $x^i \to x'^i(x^j)$ transforms as follows[1]

$$\phi'^{k_1\ldots k_p}_{l_1\ldots l_q}(x') = \frac{\partial x'^{k_1}}{\partial x^{i_1}}\cdots\frac{\partial x'^{k_p}}{\partial x^{i_p}}\,\frac{\partial x^{j_1}}{\partial x'^{l_1}}\cdots\frac{\partial x^{j_q}}{\partial x'^{l_q}}\;\phi^{i_1\ldots i_p}_{j_1\ldots j_q}(x)\,.$$

Here tensor indices are acted with the matrices $\frac{\partial x'^i}{\partial x^j}$ which form a group $\mathrm{GL}(d,\mathbb{R})$. This is a group of all invertible real $d\times d$ matrices. A simplest example is a scalar field that does not carry any indices. Its transformation law under coordinate transformations is $\phi'(x') = \phi(x)$. We stress that a point with coordinates x in the original frame and a point with coordinates x' in the transformed frame is the one and the same geometric point.

2.2 Noether's theorem

In order to fully describe a dynamical system, it is not enough to only know the equations of motion. It is also important to be able to express the basic physical characteristics, in particular, the dynamical invariants, via solutions of these equations. This goal is achieved by means of the first Noether theorem, which we formulate and prove below.

Noether's first theorem: *To any finite-parametric, i.e. dependent on s constant parameters, continuous transformation of the fields and the space-time coordinates which leaves the action invariant corresponds s dynamical invariants, i.e. the conserved functions of the fields and their derivatives.*

To prove the theorem, consider an infinitezimal transformation

$$\begin{aligned} &x^i \to x'^i = x^i + \delta x^i\,, \qquad i = 1,\ldots,d,\\ &\phi_I(x) \to \phi'_I(x') = \phi_I(x) + \delta\phi_I(x)\,. \end{aligned}$$

As in the finite-dimensional case, the variations δx^i and $\delta\phi_I$ are expressed via infinitezimal linearly independent parameters $\delta\omega_n$:

$$\delta x^i = \sum_{1\leqslant n\leqslant s} X^i_n\delta\omega_n\,, \qquad \delta\phi_I(x) = \sum_{1\leqslant n\leqslant s}\Phi_{I,n}\delta\omega_n\,. \tag{2.5}$$

[1]There is a simple rule to remember the appearance of primed and unprimed indices in the tensor transformation rule. Assuming that all indices on the left hand side of the tensor transformation formula are 'primed', then they must label 'primed' coordinates in the right hand side of the formula.

Here all $\delta\omega_n$ are independent of the coordinates x. Such transformations are called *global*. The coefficients X^i_n and $\Phi_{I,n}$ may depend on x and the fields, and they describe a response of coordinates and fields on the infinitezimal transformation with a parameter $\delta\omega_n$.

Obviously, particular cases of the transformations above arise, when $X^k_n = 0$ or $\Phi_{I,n} = 0$. In the first case the coordinates x^i do not change under symmetry transformations at all, while the fields are transformed according to

$$\phi_I(x) \to \phi'_I(x) = \phi_I(x) + \delta\phi_I(x)\,.$$

In the second case the symmetry acts on the space-time coordinates only and the condition $\Phi_{I,n} = 0$ implies that $\phi'_I(x') = \phi_I(x)$, *i.e.* the fields under considerations are scalars. We point out that in the case when ϕ_I is not a scalar but rather a tensor, $\Phi_{I,n}$ is not zero even if the symmetry acts on the space-time coordinates only! To illustrate this point, consider a vector field $\phi^i(x)$. Under coordinate transformation $x^i \to x'^i = x^i + \delta x^i$ one gets

$$\phi'^i(x') = \frac{\partial x'^i}{\partial x^j}\phi^j(x) = \frac{\partial(x^i + \delta x^i)}{\partial x^j}\phi^j(x) = \phi^i(x) + \underbrace{\frac{\partial \delta x^i}{\partial x^j}\phi^j(x)}_{\delta\phi^i}\,,$$

which implies that the corresponding quantity Φ_I is non-trivial; the trivial case occurs only when δx^i does not depend on coordinates, *i.e.* it is a constant.

In the general case symmetry transformations act on both the space-time coordinates and the fields, cf. eq.(2.5). Consider

$$\phi'_I(x') = \phi'_I(x+\delta x) = \phi'_I(x) + \partial_k\phi'_I(x)\delta x^k + \ldots = \phi'_I(x) + \partial_k\phi_I(x)\,X^k_n\delta\omega_n + \ldots$$

It is important to realize that the operations δ and $\partial/\partial x$ do not commute. This is because δ is the variation of the fields due to both the change of their form and their arguments x^i. We therefore introduce the notion of the variation of the form of the field function

$$\bar\delta\phi_I(x) = \phi'_I(x) - \phi_I(x) = (\Phi_{I,n} - \partial_k\phi_I\,X^k_n)\delta\omega_n\,.$$

Variation of the form does commute with the derivative $\partial/\partial x$. For the variation of the Lagrangian density we, therefore, have

$$\mathscr{L}'(x') = \mathscr{L}'(x) + \frac{d\mathscr{L}}{dx^k}\delta x^k = \mathscr{L}(x) + \underbrace{\mathscr{L}'(x) - \mathscr{L}(x)}_{\bar\delta\mathscr{L}(x)} + \frac{d\mathscr{L}}{dx^k}\delta x^k\,.$$

The change of the action is[2]

$$\delta S = \int \mathrm{d}x'\,\mathscr{L}'(x') - \int \mathrm{d}x\,\mathscr{L}(x) = \int \mathrm{d}x'\,[\mathscr{L}(x) + \bar\delta\mathscr{L}(x) + \frac{d\mathscr{L}}{dx^k}\delta x^k] - \int \mathrm{d}x\,\mathscr{L}(x)\,.$$

Transformation of the integration measure is

$$\mathrm{d}x' = J\cdot\mathrm{d}x \equiv \underbrace{\det\begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \cdots & \frac{\partial x'^d}{\partial x^1} \\ \vdots & & \vdots \\ \frac{\partial x'^1}{\partial x^d} & \cdots & \frac{\partial x'^d}{\partial x^d} \end{pmatrix}}_{\text{Jacobian}}\mathrm{d}x = \det\begin{pmatrix} 1+\frac{\partial\delta x^1}{\partial x^1} & \cdots & \frac{\partial\delta x^d}{\partial x^1} \\ \vdots & & \vdots \\ \frac{\partial\delta x^1}{\partial x^d} & \cdots & 1+\frac{\partial\delta x^d}{\partial x^d} \end{pmatrix}\mathrm{d}x\,.$$

[2]We consider a field theory in d-dimensions, so that the integration measure $\mathrm{d}x$ must be understood as $\mathrm{d}x = \mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_d \equiv \mathrm{d}^d x$.

Thus, at leading order in $\delta\omega_n$ we have

$$\mathrm{d}x' = \mathrm{d}x(1 + \partial_k \delta x^k + \ldots).$$

Plugging this into the variation of the action, we find

$$\delta S = \int \mathrm{d}x \left[\bar{\delta}\mathscr{L}(x) + \frac{d\mathscr{L}}{dx^k}\delta x^k + \partial_k \delta x^k \,\mathscr{L}\right] = \int \mathrm{d}x \left[\bar{\delta}\mathscr{L}(x) + \frac{d}{dx^k}(\mathscr{L}\delta x^k)\right].$$

We further note that

$$\bar{\delta}\mathscr{L}(x) = \frac{\partial\mathscr{L}}{\partial\phi_I}\bar{\delta}\phi_I + \frac{\partial\mathscr{L}}{\partial(\partial_k\phi_I)}\partial_k\bar{\delta}\phi_I = \partial_k\Big(\frac{\partial\mathscr{L}}{\partial(\partial_k\phi_I)}\Big)\bar{\delta}\phi_I + \frac{\partial\mathscr{L}}{\partial(\partial_k\phi_I)}\partial_k\bar{\delta}\phi_I = \\ = \partial_k\Big(\frac{\partial\mathscr{L}}{\partial(\partial_k\phi_I)}\bar{\delta}\phi_I\Big),$$

where we have used the Euler-Lagrange equations. Thus, we arrive at the following formula for the variation of the action

$$\delta S = \int \mathrm{d}x \,\frac{d}{dx^k}\Big[\frac{\partial\mathscr{L}}{\partial(\partial_k\phi_I)}\bar{\delta}\phi_I + \mathscr{L}\delta x^k\Big] = \int \mathrm{d}x \,\frac{d}{dx^k}\Big[\frac{\partial\mathscr{L}}{\partial(\partial_k\phi_I)}(\Phi_{I,n} - \partial_m\phi_I\, X_n^m) + \mathscr{L}X_n^k\Big]\delta\omega_n\,.$$

Since the integration volume is arbitrary we conclude that

$$\frac{dJ_n^k}{dx^k} = 0 \qquad \Longleftrightarrow \qquad \mathrm{div}J_n = 0\,,$$

where

$$J_n^k = -\frac{\partial\mathscr{L}}{\partial(\partial_k\phi_I)}(\Phi_{I,n} - \partial_m\phi_I\, X_n^m) - \mathscr{L}X_n^k$$

and $n = 1, \ldots s$. Thus, we have shown that the invariance of the action under the s-parametric symmetry transformations implies the existence of s conserved currents.

An important remark is in order. The quantities J_n^k are not uniquely defined. One can add

$$J_n^k \to J_n^k + \partial_m\chi_n^{km}\,,$$

where $\chi_n^{km} = -\chi_n^{mk}$. Adding such anti-symmetric functions does not influence the conservation law $\partial_k J_n^k = 0$.

Now we are ready to investigate concrete examples of symmetry transformations and derive the corresponding conserved currents.

- ***Energy-momentum tensor***. Consider the infinitezimal space-time translations

  $$x'^k = x^k + \delta x^k = x^k + \delta_n^k\delta\omega_n \qquad \Longrightarrow \qquad X_n^k = \delta_n^k$$

  and $\Phi_{I,n} = 0$. Thus, the conserved current J_n^k becomes in this case a second rank tensor T_n^k

  $$T_n^k = \frac{\partial\mathscr{L}}{\partial(\partial_k\phi_I)}\partial_n\phi_I - \delta_n^k\mathscr{L}\,.$$

  Here, as usual, the sum over the index I is assumed. The quantity T_n^k is the so-called *stress-energy* or *energy-momentum* tensor. If all the fields vanish at spacial infinity then the integral[3]

  $$P_n = \int \mathrm{d}^{n-1}x\, T_n^0$$

[3]Here we explicitly distinguished a time direction t and write the integration measure in the action as $\mathrm{d}x = \mathrm{d}t\,\mathrm{d}^{n-1}x$.

is a conserved quantity. Here 0 signifies the time direction and the integral is taken over the whole $(n-1)$-dimensional space. Indeed,

$$\frac{dP_n}{dt} = \int \mathrm{d}x \frac{dT_n^0}{dt} = -\int \mathrm{d}^{n-1}x \, \frac{dT_n^i}{dx^i} = -\int_{\Omega\to\infty} \mathrm{d}\Omega \, (\vec{T}_n \cdot \vec{n}) \,,$$

where Ω is a $(n-2)$-dimensional sphere which surrounds a $n-1$-dimensional volume; its radius tends to infinity. The vector $\vec{n}$ is a unit vector orthogonal to Ω.

- ***Angular momentum***. Consider infinitezimal rotations $x'^n \to x^n + x_m \delta\Omega^{nm}$, where $\delta\Omega^{nm} = -\delta\Omega^{mn}$. Because of anti-symmetry, we can choose $\delta\Omega^{nm} = \delta\omega^{nm}$ with $n < m$ as linearly independent transformation parameters. We find

$$\begin{aligned} \delta x^k &= X^k_j \delta\omega^j = \sum_{n<m} X^k_{nm} \delta\omega^{nm} = x_l \delta\omega^{kl} = x_l \delta^k_m \delta\omega^{ml} \\ &= \sum_{m<l} x_l \delta^k_m \delta\omega^{ml} + \sum_{m>l} x_l \delta^k_m \delta\omega^{ml} = \sum_{m<l} (x_l \delta^k_m - x_m \delta^k_l) \delta\omega^{ml} \,. \end{aligned} \tag{2.6}$$

From here we deduce that

$$X^k_{nm} = x_m \delta^k_n - x_n \delta^k_m, \quad n < m \,.$$

If we consider a scalar field then $\phi'(x') = \phi(x)$ and $\delta\phi = 0$. As a result, $\Phi_{I,n} = 0$. Using the general formula

$$J^k_n = -\frac{\partial \mathscr{L}}{\partial(\partial_k \phi_I)} (\Phi_{I,n} - \partial_m \phi_I \, X^m_n) - \mathscr{L} X^k_n \,,$$

we therefore find the following angular momentum tensor

$$M^k_{lm} = \frac{\partial \mathscr{L}}{\partial(\partial_k \phi)} (\partial_l \phi \, x_m - \partial_m \phi \, x_l) + \mathscr{L}(x_l \delta^k_m - x_m \delta^k_l) \,.$$

Notice that the last formula can be written in the form

$$M^k_{lm} = x_m \Big(\frac{\partial \mathscr{L}}{\partial(\partial_k \phi)} \partial_l \phi - \mathscr{L} \delta^k_l \Big) - x_l \Big(\frac{\partial \mathscr{L}}{\partial(\partial_k \phi)} \partial_m \phi - \mathscr{L} \delta^k_m \Big) = x_m T^k_l - x_l T^k_m \,,$$

where T^k_l is the stress-energy tensor.

If we consider now a vector field ϕ^i, then according to the discussion above, we will have

$$\delta\phi^i = \sum_{m<l} \Phi^i_{ml} \delta w^{ml} = \frac{\partial \delta x^i}{\partial x^j} \phi^j(x) = \frac{\partial}{\partial x^j} \Big(\sum_{m<l} (x_l \delta^i_m - x_m \delta^i_l) \delta\omega^{ml} \Big)$$

so that

$$\Phi^i_{ml} = (g_{jl} \delta^i_m - g_{jm} \delta^i_l) \phi^j = \phi_l \delta^i_m - \phi_m \delta^i_l \,,$$

where g_{ij} is a space-time metric. According to our general formula, the set of corresponding Noether currents will have the form

$$J^k_{mn} = -\frac{\partial \mathscr{L}}{\partial(\partial_k \phi^i)} (\Phi^i_{mn} - \partial_l \phi^i X^l_{mn}) - \mathscr{L} X^k_{mn} \,.$$

Substitution of all the quantities gives

$$J^k_{mn} = -\frac{\partial \mathscr{L}}{\partial(\partial_k \phi^i)} \big[\phi_n \delta^i_m - \phi_m \delta^i_n - \partial_l \phi^i (x_n \delta^l_m - x_m \delta^l_n) \big] - \mathscr{L}(x_n \delta^k_m - x_m \delta^k_n) \,.$$

We, therefore, see that for the vector field, the angular-momentum tensor takes the form

$$J^k_{mn} = x_n T^k_m - x_m T^k_n - \Big(\frac{\partial \mathscr{L}}{\partial(\partial_k \phi^n)} \phi_m - \frac{\partial \mathscr{L}}{\partial(\partial_k \phi^m)} \phi_n \Big) .$$

The first piece here, which depends on the stress-energy tensor is called *the orbital momentum* and the second piece characterizes polarization properties of the field and is related with a notion of *spin*.

The final remark concern continuous s-parametric transformations which leave the action invariant up to a total derivative term (in the original formulation of the Noether's an exact invariance of the action was assumed!)

$$\delta S = \delta \omega_n \int \mathrm{d}x \, \partial_k F^k_n \, .$$

These transformations also lead to conservation laws. It obtain them, it is enough to subtract from the canonical current J^k_n the term F^k_n:

$$\mathscr{J}^k_n = J^k_n - F^k_n \, .$$

One can verify that this new current is conserved $\partial_k \mathscr{J}^k_n$ as the consequence of the equations of motion.

Chapter 3

Klein-Gordon field

> The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction.
>
> *Sidney Coleman*

3.1 Classical Klein-Gordon field

Classical fields are functions on space-time parametrized by coordinates $x^\mu = (ct, \vec{x})$, where c is the speed of light. In application to particle physics we will consider Lorentz invariant theories, the simplest of them being a theory of a scalar field $\phi(\vec{x}, t)$. In these lectures we assume the Minkowski metric to have the signature $(1, -1, \ldots, -1)$. The action describing a free massive scalar field in four-dimensional space-time has the form

$$S[\phi] = \tfrac{1}{c} \int \mathrm{d}^4x \Big[\tfrac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \tfrac{1}{2} \big(\tfrac{mc}{\hbar}\big)^2 \phi^2(x) \Big] , \tag{3.1}$$

where $\hbar$ is the Planck constant and $\mathrm{d}^4x = c\,\mathrm{d}t\,\mathrm{d}\vec{x}$ is the Lorentz invariant integration measure. The Euler-Lagrange equation which follows from this action is called the Klein-Gordon equation[1]

$$\Big(\partial_\mu \partial^\mu + \big(\tfrac{mc}{\hbar}\big)^2 \Big) \phi(x) = 0 \, . \tag{3.2}$$

This is the equation of motion for the scalar field.

We can rewrite the action as

$$S[\phi] = \int \mathrm{d}t \mathrm{d}\vec{x} \Big[\tfrac{1}{2c^2} \dot{\phi}(x)^2 - \tfrac{1}{2} (\vec{\nabla} \phi(x))^2 - \tfrac{1}{2} \big(\tfrac{mc}{\hbar}\big)^2 \phi^2(x) \Big] = \int \mathrm{d}t \, L \, , \tag{3.3}$$

where L is identified with the corresponding Lagrangian and $\dot{\phi} = \partial_t \phi$.

[1] Very often in quantum field theory one adopts the natural units $\hbar = 1 = c$. In the international system of units SI the Klein-Gordon operator reads as

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_i^2} + \frac{m^2 c^2}{\hbar^2} \, .$$

We point out that $\lambda = \frac{\hbar}{mc}$ is the (reduced) Compton wave length associated to the scalar field.

The passage to the Hamiltonian formalism is performed by introducing the canonical momentum $\pi(x)$ conjugate to the "coordinate" $\phi(x)$:

$$\pi(x) = \frac{\delta L}{\delta \dot\phi(x)} = \frac{\dot\phi(x)}{c^2}\,.$$

The Hamiltonian has the form

$$H = \int \mathrm{d}\vec{x}\, \pi\dot\phi - L\,,$$

where in the right hand side of the last formula one has to substitute the expression for $\dot\phi(x)$ via $\pi(x)$. Making this substitution, one obtaines the Hamiltonian of the Klein-Gordon field

$$H = \frac{1}{2}\int \mathrm{d}\vec{x}\left[c^2\pi^2 + \partial_i\phi\,\partial_i\phi + \left(\tfrac{mc}{\hbar}\right)^2\phi^2\right]. \tag{3.4}$$

Denote by $[\phi]$ the physical dimension of the field ϕ. Since H has the dimension of energy $[H] = \mathscr{E}$, we have $\mathscr{E} = \ell^3 \times \frac{1}{\ell^2} \times [\phi]^2$, where $\mathscr{E}$ and ℓ signify units of energy and length, respectively. This shows that the physical dimensions[2] of a relativistic scalar field and of its momentum are

$$[\phi] = \sqrt{\frac{\mathscr{E}}{\ell}}\,, \qquad [\pi] = \frac{1}{c\ell}\sqrt{\frac{\mathscr{E}}{\ell}} = \sqrt{\frac{m}{\ell^3}}\,.$$

Looking at the formula for the action, it is not difficult to deduce the physical dimension of the latter; it is

$$[S] = \frac{1}{c}\times\ell^4\times\frac{1}{\ell^2}\times[\phi]^2 = \frac{\ell}{c}\times\mathscr{E} = \mathscr{E}\times t = [\hbar]\,.$$

Thus, the physical dimension of the action coincides with that of the Planck constant $\hbar$. Concerning the dimension of $\hbar$, it is

$$[\hbar] = \text{energy}\times\text{time} = \text{momentum}\times\text{coordinate} = \text{angular momentum}\,.$$

The definition of the Poisson brackets is also generalized to the field-theoretic case. For any two local in time functionals $F[\pi,\phi]$ and $G[\pi,\phi]$ of fields and their momenta we define their Poisson bracket as the following functional

$$\{F,G\} = \int \mathrm{d}\vec{x}\left[\frac{\delta F}{\delta\pi(x)}\frac{\delta G}{\delta\phi(x)} - \frac{\delta G}{\delta\pi(x)}\frac{\delta F}{\delta\phi(x)}\right],$$

where F and G are taken at the same moment of time. Let us show that just as in classical mechanics with a finite number of degrees of freedom the physical dimension of the Poisson bracket, which we denote by $[\{F,G\}]$, is offset from the product of physical dimensions $[F][G]$ by one power of $\hbar$. To this end, we need to find the dimensions of the variational derivatives entering the formula for the Poisson bracket. Using the definition of the variational derivative, we find

$$\begin{aligned}
\delta F &= \int \mathrm{d}\vec{x}\,\frac{\delta F}{\delta\phi(x)}\delta\phi(x) \quad\longrightarrow\quad [F] = \ell^3\times\left[\frac{\delta F}{\delta\phi(x)}\right]\times\sqrt{\frac{\mathscr{E}}{\ell}} \quad\longrightarrow\quad \left[\frac{\delta F}{\delta\phi(x)}\right] = \frac{[F]}{\ell^{5/2}\times\mathscr{E}^{1/2}}\,,\\
\delta F &= \int \mathrm{d}\vec{x}\,\frac{\delta F}{\delta\pi(x)}\delta\pi(x) \quad\longrightarrow\quad [F] = \ell^3\times\left[\frac{\delta F}{\delta\pi(x)}\right]\times\sqrt{\frac{\mathscr{E}}{c^2\ell^3}} \quad\longrightarrow\quad \left[\frac{\delta F}{\delta\pi(x)}\right] = \frac{[F]\times c}{\ell^{3/2}\times\mathscr{E}^{1/2}}\,.
\end{aligned}$$

[2]Physical dimension is often called 'the engineering dimension', see *e.g.* "Quantum Fields and Strings: A Course for Mathematicians" AMS IAS 2000, by Perre Deligne et al., vol 1, page 446 (remark 1).

Then from the definition of the Poisson bracket we can read off its physical dimension

$$\big[\{F,G\}\big] = \ell^3 \times \frac{[F]\times[G]\times c}{\ell^4 \times \mathscr{E}} = \frac{[F]\times[G]\times c}{\ell \times \mathscr{E}} = \frac{[F]\times[G]}{[\hbar]}\,. \tag{3.5}$$

The canonical Poisson brackets implied by the Lagrangian (3.3) are

$$\begin{aligned} \{\phi(t,\vec{x}),\phi(t,\vec{y})\} &= 0\,,\\ \{\pi(t,\vec{x}),\pi(t,\vec{y})\} &= 0\,,\\ \{\pi(t,\vec{x}),\phi(t,\vec{y})\} &= \delta(\vec{x}-\vec{y})\,. \end{aligned} \tag{3.6}$$

Note that all the fields for which the brackets are computed are taken at the one and the same moment of time. Then the Hamiltonian equations can be cast in the following form

$$\dot{\phi} = \{H,\phi\}\,, \qquad\qquad \dot{\pi} = \{H,\pi\}\,.$$

The first equation here gives $\dot{\phi} = c^2\pi$, while the second one results in $\dot{\pi} = \partial_i^2\phi - \left(\frac{mc}{\hbar}\right)^2\phi$. Differentiating the first equation over time, we then get

$$\ddot{\phi} = c^2\dot{\pi} = c^2\Big[\partial_i^2\phi - \Big(\frac{mc}{\hbar}\Big)^2\phi\Big]\,, \tag{3.7}$$

which is equivalent to (3.2).

It appears that the most efficient way to look at the dynamical variables is to invoke a momentum representation, which simply means a passage to the corresponding Fourier image

$$\phi(x) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^4k\, e^{ikx}\tilde{\phi}(k)\,. \tag{3.8}$$

Here the integration measure is $\mathrm{d}^4k = \mathrm{d}k^0\mathrm{d}k^1\ldots\mathrm{d}k^3$ and the Lorentz invariant scalar product kx is defined as

$$kx \equiv k_\mu x^\mu = k^0x^0 - \vec{k}\,\vec{x} = \omega t - \vec{k}\,\vec{x}\,,$$

where $\vec{k}$ is the wave vector, ω is the frequency and

$$x^0 \equiv ct\,, \qquad k^0 = \frac{\omega}{c}\,.$$

In what follows we call the Fourier transform (3.8) the *wave type*, as the basis functions over which the expansion is made of are the standard plane-waves $e^{ikx} = e^{i(\omega t - \vec{k}\vec{x})}$.

Quite often one uses another, the *energetic-type* representation. It is based on the de Broglie formulae

$$E = \hbar\omega\,, \qquad \vec{p} = \hbar\vec{k}\,,$$

where E and $\vec{p}$ are the energy and momentum of the wave, respectively. Making the corresponding change of variables, we get the energetic-type Fourier transform

$$\phi(x) = \int \frac{\mathrm{d}E\mathrm{d}\vec{p}}{(2\pi)^{3/2}c\hbar^4}\, e^{\frac{i}{\hbar}(Et-\vec{p}\vec{x})}\tilde{\phi}(p)\,. \tag{3.9}$$

The Klein-Gordon equation for the Fourier image takes the form

$$\Big[\big(\tfrac{E}{c}\big)^2 - \vec{p}^{\,2} - m^2c^2\Big]\tilde{\phi}(p) = 0\,. \tag{3.10}$$

Since $p^\mu = \left(\frac{E}{c}, \vec{p}\right)$ is the four-momentum, the last equation written in the relativistic invariant form reads $\left(p_\mu p^\mu - m^2c^2\right)\tilde{\phi}(p) = 0$ and it is solved by

$$\tilde{\phi}(p) = \delta\left(p^2 - m^2c^2\right)\varphi(p)\,.$$

The multiplier $\delta(p^2 - m^2c^2)$ establishes a relation between energy variable E and momentum $\vec{p}$ and the square of mass m^2

$$p^2 - m^2c^2 = \left(\tfrac{E}{c}\right)^2 - \vec{p}^{\,2} - m^2c^2 = 0\,. \tag{3.11}$$

This relation is known as the *mass-shell condition*. Therefore,

$$\phi(x) = \int \tfrac{\mathrm{d}E\mathrm{d}\vec{p}}{(2\pi)^{3/2}c\hbar^4}\,\delta\left(p^2 - m^2c^2\right)e^{\frac{i}{\hbar}(Et - \vec{p}\vec{x})}\varphi(p)\,.$$

Because of the δ-function the integration goes over two three-dimensional hyperboloids

$$E = \pm c\sqrt{\vec{p}^2 + m^2c^2}\,.$$

Using the property of the δ-function

$$\delta\left(p^2 - m^2c^2\right) = \tfrac{c\,\delta(E - c\sqrt{\vec{p}^2 + m^2c^2})}{2\sqrt{\vec{p}^2 + m^2c^2}} + \tfrac{c\,\delta(E + c\sqrt{\vec{p}^2 + m^2c^2})}{2\sqrt{\vec{p}^2 + m^2c^2}}\,,$$

the solution naturally splits into two integrals

$$\phi(x) = \tfrac{c}{(2\pi)^{3/2}\hbar^4}\left[\int \tfrac{\mathrm{d}\vec{p}}{2E}\,e^{\frac{i}{\hbar}(Et - \vec{p}\vec{x})}\,\varphi(E, \vec{p}) + \int \tfrac{\mathrm{d}\vec{p}}{2E}\,e^{-\frac{i}{\hbar}(Et + \vec{p}\vec{x})}\,\varphi(-E, \vec{p})\right],$$

where now $E = c\sqrt{\vec{p}^2 + m^2c^2}$ is assumed to be always positive! The second integrand is not written in the relativistic-invariant from, so we change $\vec{p} \to -\vec{p}$ obtaining thereby

$$\phi(x) = \tfrac{c}{(2\pi)^{3/2}\hbar^4}\left[\int \tfrac{\mathrm{d}\vec{p}}{2E}\,e^{\frac{i}{\hbar}(Et - \vec{p}\vec{x})}\,\varphi(E, \vec{p}) + \int \tfrac{\mathrm{d}\vec{p}}{2E}\,e^{-\frac{i}{\hbar}(Et - \vec{p}\vec{x})}\,\varphi(-E, -\vec{p})\right],$$

The first term in the brackets called the positive frequency part of $\phi(x)$, while the second one the negative frequency part, respectively. It is standard to introduce the following Fourier amplitudes

$$a^*(\vec{p}) = \tfrac{\varphi(E,\vec{p})}{\hbar^3\sqrt{2E}}\,, \qquad a(\vec{p}) = \tfrac{\varphi(-E,-\vec{p})}{\hbar^3\sqrt{2E}}\,.$$

For a real scalar field $\varphi^*(p) = \varphi(-p)$, so that $a(\vec{p})$ and $a^*(\vec{p})$ are in fact complex conjugate to each other. The dimension of $\varphi(p)$ is $[\varphi(p)] = \hbar^2\ell^{3/2}\mathscr{E}^{1/2}$, so that the dimension of $a(\vec{p})$ is $[a(\vec{p})] = \hbar^{1/2}p^{-3/2}$. With the introduction of these amplitudes, the corresponding Fourier expansions will finally read

$$\phi(x) \;=\; c\hbar^{1/2}\int \frac{\mathrm{d}\vec{p}}{(2\pi\hbar)^{3/2}}\,\frac{1}{\sqrt{2E}}\Big[a^*(\vec{p})\,e^{\frac{i}{\hbar}(Et - \vec{p}\vec{x})} + a(\vec{p})\,e^{-\frac{i}{\hbar}(Et - \vec{p}\vec{x})}\Big]\,, \tag{3.12}$$

$$\pi(x) \;=\; \frac{i}{2c\hbar^{1/2}}\int \frac{\mathrm{d}\vec{p}}{(2\pi\hbar)^{3/2}}\,\sqrt{2E}\Big[a^*(\vec{p})\,e^{\frac{i}{\hbar}(Et - \vec{p}\vec{x})} - a(\vec{p})\,e^{-\frac{i}{\hbar}(Et - \vec{p}\vec{x})}\Big]\,. \tag{3.13}$$

Let us now express the Hamiltonian (3.4) in terms of the amplitudes $a(\vec{p})$ and $a^*(\vec{p})$. Substitution of $\phi(x)$ and $\pi(x)$ would lead to a triple integral, but the x-integration is easily done yielding the delta-function, which allows one to further perform one of the momentum integrals. We recall the Fourier representation of the the Dirac delta-function

$$\delta(\vec{p}) = \int \frac{\mathrm{d}\vec{x}}{(2\pi\hbar)^3}\, e^{\frac{i}{\hbar}\vec{p}\vec{x}}\,. \tag{3.14}$$

In view of the importance of this calculation we will perform it here in full detail. We start with

$$H = \frac{1}{2}\int \mathrm{d}\vec{x}\left(c^2\pi^2 + \partial_i\phi\,\partial_i\phi + \left(\tfrac{mc}{\hbar}\right)^2\phi^2\right).$$

We then get

$$\begin{aligned}
H \;=\; & \frac{1}{2\hbar}\int \frac{\mathrm{d}\vec{x}\,\mathrm{d}\vec{p}\,\mathrm{d}\vec{p}'}{(2\pi\hbar)^3}\Big[-\frac{1}{4}\sqrt{4EE'}\Big(a^*(\vec{p})a^*(\vec{p}')e^{\frac{i}{\hbar}(E+E')t-\frac{i}{\hbar}(\vec{p}+\vec{p}')\vec{x}} - a^*(\vec{p})a(\vec{p}')e^{\frac{i}{\hbar}(E-E')t-\frac{i}{\hbar}(\vec{p}-\vec{p}')\vec{x}} \\
& \qquad -a(\vec{p})a^*(\vec{p}')e^{-\frac{i}{\hbar}(E-E')t+\frac{i}{\hbar}(\vec{p}-\vec{p}')\vec{x}} + a(\vec{p})a(\vec{p}')e^{-\frac{i}{\hbar}(E+E')t+\frac{i}{\hbar}(\vec{p}+\vec{p}')\vec{x}}\Big) \\
& + \frac{c^2}{\sqrt{4EE'}}\Big(-(\vec{p}\vec{p}')a^*(\vec{p})a^*(\vec{p}')e^{\frac{i}{\hbar}(E+E')t-\frac{i}{\hbar}(\vec{p}+\vec{p}')\vec{x}} + (\vec{p}\vec{p}')a^*(\vec{p})a(\vec{p}')e^{\frac{i}{\hbar}(E-E')t-\frac{i}{\hbar}(\vec{p}-\vec{p}')\vec{x}} \\
& \qquad +(\vec{p}\vec{p}')a(\vec{p})a^*(\vec{p}')e^{-\frac{i}{\hbar}(E-E')t+\frac{i}{\hbar}(\vec{p}-\vec{p}')\vec{x}} - (\vec{p}\vec{p}')a(\vec{p})a(\vec{p}')e^{-\frac{i}{\hbar}(E+E')t+\frac{i}{\hbar}(\vec{p}+\vec{p}')\vec{x}}\Big) \\
& + \frac{m^2c^4}{\sqrt{4EE'}}\Big(a^*(\vec{p})a^*(\vec{p}')e^{\frac{i}{\hbar}(E+E')t-\frac{i}{\hbar}(\vec{p}+\vec{p}')\vec{x}} + a^*(\vec{p})a(\vec{p}')e^{\frac{i}{\hbar}(E-E')t-\frac{i}{\hbar}(\vec{p}-\vec{p}')\vec{x}} \\
& \qquad +a(\vec{p})a^*(\vec{p}')e^{-\frac{i}{\hbar}(E-E')t+\frac{i}{\hbar}(\vec{p}-\vec{p}')\vec{x}} + a(\vec{p})a(\vec{p}')e^{-\frac{i}{\hbar}(E+E')t+\frac{i}{\hbar}(\vec{p}+\vec{p}')\vec{x}}\Big)\,.
\end{aligned}$$

Integrating over x produces either $\delta(\vec{p}-\vec{p}')$ or $\delta(p+\vec{p}')$. Thus, we can further integrate over $\vec{p}'$ obtaining the following result

$$\begin{aligned}
H \;=\; & \frac{1}{2\hbar}\int \mathrm{d}\vec{p}\Big[-\tfrac{1}{2}E\Big(a^*(\vec{p})a^*(-\vec{p})e^{\frac{2i}{\hbar}Et} - 2a^*(\vec{p})a(\vec{p}) + a(\vec{p})a(-\vec{p})e^{-\frac{2i}{\hbar}Et}\Big) \\
& \quad + \tfrac{\vec{p}^2c^2}{2E}\Big(a^*(\vec{p})a^*(-\vec{p})e^{\frac{2i}{\hbar}Et} + 2a^*(\vec{p})a(\vec{p}) + a(\vec{p})a(-\vec{p})e^{-\frac{2i}{\hbar}Et}\Big) \\
& \quad + \tfrac{m^2c^4}{2E}\Big(a^*(\vec{p})a^*(-\vec{p})e^{\frac{2i}{\hbar}Et} + 2a^*(\vec{p})a(\vec{p}) + a(\vec{p})a(-\vec{p})e^{-\frac{2i}{\hbar}Et}\Big)\Big]\,.
\end{aligned}$$

Thus, combining similar terms we arrive at

$$\begin{aligned}
H \;=\; & \int \frac{\mathrm{d}\vec{p}}{4\hbar E}\Big[\Big(-E^2 + \vec{p}^2c^2 + m^2c^4\Big)\Big(a^*(\vec{p})a^*(-\vec{p})e^{\frac{2i}{\hbar}Et} + a(\vec{p})a(-\vec{p})e^{-\frac{2i}{\hbar}Et}\Big) + \\
& \quad 2\Big(E^2 + \vec{p}^2c^2 + m^2c^4\Big)a^*(\vec{p})a(\vec{p})\Big]\,.
\end{aligned}$$

The first line in the expression above vanishes due to the fact that $E(\vec{p})^2 = \vec{p}^2c^2 + m^2c^4$, the second line gives

$$H = \frac{1}{\hbar}\int \mathrm{d}\vec{p}\,E(\vec{p})\,a^*(\vec{p})a(\vec{p})\,.$$

Thus, written in terms of the amplitudes $a(\vec{p})$ and $a^*(\vec{p})$ the Hamiltonian is

$$H = \frac{1}{\hbar}\int \mathrm{d}\vec{p}\,E(\vec{p})\,a^*(\vec{p})a(\vec{p})\,. \tag{3.15}$$

Here $E(\vec{p}) = \sqrt{\vec{p}^2c^2 + m^2c^4}$. In general, $E(\vec{p})$ is called the dispersion relation – an expression which renders how *the energy of a single particle state depends on its momentum.* The Hamiltonian is real and manifestly positive. It might seem strange that the classical Hamiltonian contains the inverse Planck constant in front, but the latter is there for dimensional reasons – we set up to write the Hamiltonian as an integral over the three-momentum, and the Planck constant is needed to compensate for the engineering dimensions of a and a^*, which are $[a] = [a^*] = \hbar^{1/2}p^{-3/2}$. The Hamiltonian has the dimension of energy and when being written in terms of the corresponding frequency $\omega(\vec{p})$ it takes a more familiar form

$$H = \int \mathrm{d}\vec{p}\,\omega(\vec{p})a^*(\vec{p})a(\vec{p})\,. \tag{3.16}$$

For completeness we give here the expression for $\phi(x)$ in terms of frequencies

$$\phi(x) \quad = \quad c\int \frac{\mathrm{d}\vec{p}}{(2\pi\hbar)^{3/2}}\,\frac{1}{\sqrt{2\omega(p)}}\Big[a^*(\vec{p})\,e^{i(\omega t-\vec{k}\vec{x})}+a(\vec{p})\,e^{-i(\omega t-\vec{k}\vec{x})}\Big]\,. \tag{3.17}$$

Further, we note that it naturally splits into a sum of two parts $\phi = \phi^+ + \phi^-$. Here ϕ^+, being a positive-frequency solution, depends on the amplitude $a^*(\vec{p})$. Analogously, a negative-frequency solution ϕ^- involves $a(\vec{p})$:

$$\phi^+(x) = c\int \frac{\mathrm{d}\vec{p}}{(2\pi\hbar)^{3/2}}\,\frac{e^{i(\omega t-\vec{k}\vec{x})}}{\sqrt{2\omega(p)}}a^*(\vec{p})\,, \qquad \phi^-(x) = c\int \frac{\mathrm{d}\vec{p}}{(2\pi\hbar)^{3/2}}\,\frac{e^{-i(\omega t-\vec{k}\vec{x})}}{\sqrt{2\omega(p)}}a(\vec{p})\,.$$

As we will see in the next section, in quantum theory $a^*(\vec{p})$ will become an operator which creates a particle with momentum $\vec{p}$, while $a(\vec{p})$ destroys it. This makes our identification of positive frequency field with 'creation' and negative frequency field with 'annihilation' rather intuitive. We point, however, that in the literature the terminology 'positive and negative frequency' is often assigned in an opposite way: positive $\leftrightarrow$ annihilates and negative $\leftrightarrow$ creates. Such a terminology originates from intrpreting the exponential factor $e^{\mp i\omega t} = e^{\mp\frac{i}{\hbar}Et}$ from the point of view of the one-particle Schrödinger equation. Indeed, $e^{-i\omega t}$ has a positive frequency (energy), while $e^{+i\omega t}$ has a negative frequency (energy), because

$$i\hbar\frac{\partial}{\partial t}e^{\mp i\omega t} = \pm\hbar\omega e^{\mp i\omega t} = \pm E e^{\mp i\omega t}\,.$$

We will not pursue this interpretation here.[3]

So far we are in the framework of the classical theory and it is now the time to ask what is the physical meaning of the amplitudes $a^*(\vec{p})$ and $a(\vec{p})$. Looking for the expression for $\phi(x)$, we see that it is natural to define the time-dependent amplitudes as

$$a^*(\vec{p},t) = e^{i\omega(\vec{p})t}a^*(\vec{p})\,, \qquad a(\vec{p},t) = e^{-i\omega(\vec{p})t}a(\vec{p})$$

These formulae represent the solutions of the Hamiltonian equations of motion

$$\frac{da(t)}{dt} = \{H, a(t)\}\,, \qquad \frac{da^*(t)}{dt} = \{H, a^*(t)\}\,,$$

where the Poisson brackets are

$$\begin{aligned} \{a(\vec{p}), a^*(\vec{p}')\} &= i\,\delta(\vec{p}-\vec{p}')\,, \\ \{a^*(\vec{p}), a^*(\vec{p}')\} &= 0\,, \\ \{a(\vec{p}), a(\vec{p}')\} &= 0\,. \end{aligned} \tag{3.18}$$

It is easy to check that these Poisson brackets imply the Poisson brackets (3.6) for $\phi(\vec{x})$ and $\pi(\vec{x})$. Thus, we are led to conclude that the free massive scalar field is nothing but an infinite set of harmonic oscillators. Indeed, the Hamiltonian (3.16) is essentially a sum of frequencies of an infinite number of harmonic oscillators, each labeled by the three-dimensional momentum vector $\vec{p}$. Often representation (3.12) and (3.13) of canonical fields in terms of the complex amplitudes a and a^* is called *holomorphic*.

Since the action for the Klein-Gordon field is invariant under Poincaré group, we can use the Noether theorem to construct the corresponding Noether currents and the conserved charges, among

[3] Our definition of positive and negative frequency solutions agrees with that of Bogoluibov & Shirkov but opposite to Peskin & Schroeder.

them, in addition to the Hamiltonian, the momentum P_i and rotations J_{ij} and Lorentz boosts J_{0i}. Explicitly, the time components of the stress-tensor are

$$T^0_0 = \mathscr{H}\,, \qquad T^0_i = c\,\pi\partial_i\phi \equiv c\mathscr{P}_i\,,$$

where $\mathscr{H}$ and $\mathscr{P}_i$ are the Hamiltonian and momentum density, respectively. The time component of the generator of Lorentz transformations is

$$J^0_{\mu\nu} = x_\mu T^0_\nu - x_\nu T^0_\mu\,.$$

Thus, we find

- Shifts
$$P^i = -P_i = -\int \mathrm{d}\vec{x}\,\pi\partial_i\phi = \frac{1}{\hbar}\int \mathrm{d}\vec{p}\,a^*(\vec{p})\,p^i a(\vec{p})\,;$$

- Rotations
$$J^{ij} = J_{ij} = \int \mathrm{d}\vec{x}\,\pi(x_i\partial_j\phi - x_j\partial_i\phi) = i\int \mathrm{d}\vec{p}\,a^*(\vec{p})(p_j\partial_i - p_i\partial_j)a(\vec{p})\,;$$

- Lorentz boosts
$$\begin{aligned} J^{0i} = -J^{i0} &= \int \mathrm{d}\vec{x}\left(\frac{1}{c}x^i\mathscr{H} - x^0\mathscr{P}^i\right) = \frac{1}{c}\int \mathrm{d}\vec{x}\,x^i\mathscr{H} - ctP^i = \\ &= \frac{i}{2c}\int \mathrm{d}\vec{p}\,E(\vec{p})\Big(a^*(\vec{p})\partial_i a(\vec{p}) - \partial_i a^*(\vec{p})a(\vec{p})\Big) - ctP^i\,. \end{aligned}$$

Normalization of generators is chosen in such a way that their engineering dimensions are of momentum for P^i and of angular momentum (of the action) for $J^{\mu\nu}$. Note that Lorentz boosts J^{0i} have an explicit time dependence. This is a manifestation of the fact that in the Hamiltonian formulation the boost symmetries are 'broken', while the Hamiltonian is one of the generators of the Poincaré algebra. The time derivative J^{0i} is

$$\frac{dJ^{0i}}{dt} = \{H, J^{0i}\} + \frac{\partial J^{0i}}{\partial t} = \{H, J^{0i}\} - c\,P^i$$

and it vanishes due to the Poincaré algebra relation $\{H, J^{0i}\} = c\,P^i$, as the reader can verify by direct calculation.

3.2 Canonical quantization and Fock space

Upon quantization the classical fields $\phi(x)$ and $\pi(x)$ become hermitian operator-valued functions[4] on space-time and they constitute observables in quantum field theory. It is important to realize that $\phi(x)$ and $\phi(x')$, which is the same operator ϕ but evaluated at two different space-time points, define *two different* observables. Canonical quantization consists in replacing the equal-time Poisson brackets $\{\ ,\ \}$ with the quantum Poisson brackets $\{\ ,\ \}_\hbar$. Explicitly,

$$\begin{aligned} \{\phi(t,\vec{x}),\phi(t,\vec{y})\}_\hbar &= \frac{i}{\hbar}[\phi(t,\vec{x}),\phi(t,\vec{y})] = 0\,, \\ \{\pi(t,\vec{x}),\pi(t,\vec{y})\}_\hbar &= \frac{i}{\hbar}[\pi(t,\vec{x}),\pi(t,\vec{y})] = 0\,, \\ \{\pi(t,\vec{x}),\phi(t,\vec{y})\}_\hbar &= \frac{i}{\hbar}[\pi(t,\vec{x}),\phi(t,\vec{y})] = \delta(\vec{x}-\vec{y})\,, \end{aligned} \tag{3.19}$$

[4] More precisely, operator-valued distributions.

which is equivalent to the following commutation relations

$$\begin{aligned}
&[\phi(t,\vec{x}),\phi(t,\vec{y})] = 0\,,\\
&[\pi(t,\vec{x}),\pi(t,\vec{y})] = 0\,,\\
&[\phi(t,\vec{x}),\pi(t,\vec{y})] = i\hbar\,\delta(\vec{x}-\vec{y})\,.
\end{aligned} \tag{3.20}$$

Upon quantization the classical amplitudes $a^*(\vec{p})$ and $a(\vec{p})$ are replaced by operators $a^\dagger(\vec{p})$ and $a(\vec{p})$, which are creation and annihilation operators for the harmonic oscillator labeled by the momentum $\vec{p}$. The commutation relations are

$$[a(\vec{p}),a(\vec{p}')] = 0\,,\qquad [a^\dagger(\vec{p}),a^\dagger(\vec{p}')] = 0\,,\qquad [a(\vec{p}),a^\dagger(\vec{p}')] = \hbar\,\delta(\vec{p}-\vec{p}')\,. \tag{3.21}$$

The commutation relations are obviously compatible with the engineering dimensions of the oscillators being $[a] = [a^\dagger] = \hbar^{1/2}p^{-3/2}$.

Commutation relations (3.21) are the algebraic relations between the abstract operators $a(\vec{p})$ and $a^\dagger(\vec{p})$. Now we would like to realize these operators through their action on states in a suitable Hilbert space. To this end, consider a state $|0\rangle$ which we identify with a vacuum state, that is the state without particles. Acting on this state with $a^\dagger(\vec{p})$ will create a one-particle state

$$|\vec{p}\rangle \equiv \sqrt{2\omega(p)}\,a^\dagger(\vec{p})|0\rangle$$

and so on

$$|0\rangle\,,\quad |\vec{p}_1\rangle,\quad |\vec{p}_1\,\vec{p}_2\rangle,\quad \dots\quad |\vec{p}_1\,\vec{p}_2\dots\vec{p}_n\rangle,\quad \dots$$

The additional normalization factor $\sqrt{2\omega(p)}$ is chosen here for later convenience[5]. The space of all these states is called the Fock space and the representation of the field operators in the Fock space is usually called *the representation of second quantization*.[6] For the discussion of the formalism of second quantization in the non-relativistic quantum mechanics we refer the reader to appendix 3.5.1.

In this space creation operator $a^\dagger(\vec{p})$ simply adds up a new particle with momentum $\vec{p}$

$$a^\dagger(\vec{p})|\vec{p}_1\,\vec{p}_2\dots\vec{p}_n\rangle = \frac{1}{\sqrt{2\omega(p)}}|\vec{p}\,\vec{p}_1\,\vec{p}_2\dots\vec{p}_n\rangle\,.$$

The annihilation operator acts in this representation as

$$a(\vec{p})|\vec{p}_1\,\vec{p}_2\dots\vec{p}_n\rangle = \hbar\sqrt{2\omega(p)}\sum_{i=1}^{n}\delta(\vec{p}-\vec{p}_i)|\vec{p}_1\ \dots\hat{\vec{p}}_i\dots\vec{p}_n\rangle\,,\qquad a(p)|0\rangle = 0\,.$$

In other words, the annihilation operator checks an existence in a state a particle with momentum $\vec{p}$ and then removes it. It is easy to verify that the formulae above indeed provide a representation of the commutation relations (3.21). Indeed,

$$\begin{aligned}
a(\vec{p})a^\dagger(\vec{p}')\,|\vec{p}_1\ \dots\vec{p}_n\rangle &= \frac{1}{\sqrt{2\omega(p')}}a(p)|\vec{p}'\ \vec{p}_1\ \dots\vec{p}_n\rangle =\\
&= \hbar\,\delta(\vec{p}-\vec{p}')|\vec{p}_1\ \dots\vec{p}_n\rangle + \hbar\sqrt{\frac{\omega(p)}{\omega(p')}}\sum_{i=1}^{n}\delta(\vec{p}-\vec{p}_i)|\vec{p}'\,\vec{p}_1\ \dots\hat{\vec{p}}_i\dots\vec{p}_n\rangle\,,\\
a^\dagger(\vec{p}')a(\vec{p})\,|\vec{p}_1\ \dots\vec{p}_n\rangle &= \hbar\sqrt{2\omega(p)}\sum_{i=1}^{n}\delta(\vec{p}-\vec{p}_i)a^\dagger(\vec{p}')\,|\vec{p}_1\ \dots\hat{\vec{p}}_i\dots\vec{p}_n\rangle =\\
&= \hbar\sqrt{\frac{\omega(p)}{\omega(p')}}\sum_{i=1}^{n}\delta(\vec{p}-\vec{p}_i)|\vec{p}'\,\vec{p}_1\ \dots\hat{\vec{p}}_i\dots\vec{p}_n\rangle\,.
\end{aligned}$$

[5]Often in the literature (see *e.g.* Sterman 1993) the operators $a, a\dagger$ are rescaled as $\mathbf{a} = \sqrt{2\omega(p)}\,a(\vec{p})$, $\mathbf{a}^\dagger = \sqrt{2\omega(p)}\,a^\dagger(\vec{p})$ so that $\mathbf{a}$ and $\mathbf{a}^\dagger$ are conjugate to each other and $[\mathbf{a}] = [\mathbf{a}^\dagger] = \mathscr{E}^{1/2}p^{-3/2}$. In this case $|\vec{p}\rangle = \mathbf{a}^\dagger|0\rangle$, while the commutator changes to $[\mathbf{a}(\vec{p}),\mathbf{a}^\dagger(\vec{p}')] = 2\hbar\omega(\vec{p})\delta(\vec{p}-\vec{p}')$. This commutator is relativistic invariant, see the discussion below.

[6]Schweber points out (see Schweber 1994, p. 28) that the idea and procedure of second quantization goes back to Jordan, in a number of papers from 1927 (see references in Schweber 1994, pp. 695f), while the term itself was coined by Dirac.

Subtracting these two expressions leaves us with the desired result

$$[a(\vec{p}), a^\dagger(\vec{p}')]|\vec{p}_1 \ldots \vec{p}_n\rangle = \hbar\,\delta(\vec{p}-\vec{p}')|\vec{p}_1 \ldots \vec{p}_n\rangle\,.$$

An arbitrary state in the Fock space is given by a linear superposition

$$|\chi\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int \prod_{i=1}^{n} \frac{\mathrm{d}\vec{p}_i}{\sqrt{2\hbar\omega(\vec{p}_i)}}\, \chi_n(\vec{p}_1,\ldots,\vec{p}_n)|\vec{p}_1 \ldots \vec{p}_n\rangle\,, \tag{3.22}$$

where the coefficients χ_n are symmetric functions of their arguments. The scalar product in the Fock space is defined as follows. We assume that $\langle 0|0\rangle = 1$. From this assumption and commutation relations (3.21) one finds

$$\begin{aligned}\langle \vec{q}_1 \ldots \vec{q}_n|\vec{p}_1 \ldots \vec{p}_m\rangle &= \prod_{k=1}^{n}\sqrt{2\omega(\vec{p}_k)}\prod_{l=1}^{m}\sqrt{2\omega(\vec{p}_l)}\,\langle 0|a(q_1)\ldots a(q_n)a^\dagger(p_1)\ldots a^\dagger(p_m)|0\rangle = \\ &= \delta_{mn}\sum_{\mathcal{P}}\prod_{i=1}^{n} 2\hbar\omega(\vec{p}_i)\,\delta(\vec{p}_i - \vec{q}_{\mathcal{P}(i)})\,. \end{aligned}\tag{3.23}$$

Here sum is taken over all permutations $\mathcal{P}$ of the n indices $1,\ldots,n$. We recall that a permutation is an operation which correlates a set of n ordered objects, *e.g.* the set $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n$, with the same set of objects but taken in a different order. Usually such an operation, which maps $\vec{q}_1$ into $\vec{q}_{\alpha_1}$, $\vec{q}_2$ into $\vec{q}_{\alpha_2}$, etc., is denoted by

$$\mathcal{P} = \begin{pmatrix} 1 & 2 & \ldots & n \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix},$$

so that $\mathcal{P}\vec{q}_i = \vec{q}_{\alpha_i} = \vec{q}_{\mathcal{P}(i)}$. With the formula (3.23) we find a scalar product between two arbitrary states

$$\langle\Psi|\chi\rangle = \sum_{n=0}^{\infty}\int \Psi_n^*(\vec{p}_1,\ldots,\vec{p}_n)\chi_n(\vec{p}_1,\ldots,\vec{p}_n)\prod_{i=1}^{n}\mathrm{d}\vec{p}_i\,. \tag{3.24}$$

Our choice of normalization of the Fock states was deliberately chosen to render the scalar product relativistic invariant. It is enough to demonstrate this for the case of one-particle sates. On the subspace of one-particle states the scalar product (3.23) boils down to

$$\langle\vec{p}|\vec{q}\rangle = 2\hbar\omega(\vec{p})\delta(\vec{p}-\vec{q}) = 2\,cp^0\delta(\vec{p}-\vec{q})\,. \tag{3.25}$$

This scalar product is indeed relativistic invariant, *i.e.* it is invariant under Lorentz transformations

$$x'^\mu = \Lambda^\mu_\nu x^\nu\,, \qquad \eta_{\mu\nu}\Lambda^\mu_\alpha\Lambda^\nu_\beta = \eta_{\alpha\beta}\,,$$

where $\eta_{\mu\nu}$ is Minkowski metric.

To show this, we need to understand how to evaluate

$$\delta(\vec{p}'-\vec{q}') \equiv \prod_{i=1}^{3}\delta(p'^i - q'^i) = \prod_{i=1}^{3}\delta\left(\Lambda^i_\mu(p^\mu - q^\mu)\right). \tag{3.26}$$

Let φ be a function from the space of test functions and consider

$$\int\varphi(\vec{p})\prod_{i=1}^{3}\delta\left(\Lambda^i_\mu(p^\mu-q^\mu)\right)\mathrm{d}p^i = \int\varphi(\vec{p}(\vec{y}))\left|\det\left(\frac{\partial p^i}{\partial y^j}\right)\right|\prod_{i=1}^{3}\delta(y^i)\mathrm{d}y^i = \varphi(\vec{p}(0))\left|\det\left(\frac{\partial p^i}{\partial y^j}\right)\right|_{\vec{y}=0},$$

where we have performed the change of variables $p^i \to y^i = \Lambda^i_\mu (p^\mu - q^\mu)$. We compute the arising Jacobian as follows. Since

$$y^i = \Lambda^i_j (p^j - q^j) + \Lambda^i_0 (p^0 - q^0)\,,$$

we have

$$\Lambda^i_j \frac{\partial p^j}{\partial y^k} + \Lambda^i_0 \frac{\partial p^0}{\partial y^k} = \delta^i_k\,. \tag{3.27}$$

Because of the mass-shell condition $p^0 = \sqrt{m^2c^2 + \vec{p}^2}$,

$$\frac{\partial p^0}{\partial y^k} = \frac{p^j}{p^0}\frac{\partial p^j}{\partial y_k} = -\frac{p_j}{p^0}\frac{\partial p^j}{\partial y_k}$$

so that eq.(4.23) takes the form

$$\Big(\Lambda^i_j - \Lambda^i_0 \frac{p_j}{p^0}\Big)\frac{\partial p^j}{\partial y^k} = \delta^i_k\,,$$

which is nothing else but the matrix form of the product of two matrices which equals to the unit matrix. Thus, computing the determinant, we obtain

$$\det\Big(\Lambda^i_j - \Lambda^i_0 \frac{p_j}{p^0}\Big)\det\Big(\frac{\partial p^j}{\partial y^k}\Big) = 1\,.$$

Note that $\vec{y} = 0$ implies $\vec{p}(0) = \vec{q}$ and thus

$$\int \varphi(\vec{p}) \prod_{i=1}^3 \delta\Big(\Lambda^i_\mu (p^\mu - q^\mu)\Big)\mathrm{d}p^i = \frac{\varphi(\vec{q})}{\Big|\det\Big(\Lambda^i_j - \Lambda^i_0 \frac{q_j}{q^0}\Big)\Big|} = \int \mathrm{d}\vec{p}\,\varphi(\vec{p})\,\frac{\delta(\vec{p} - \vec{q})}{\Big|\det\Big(\Lambda^i_j - \Lambda^i_0 \frac{p_j}{p^0}\Big)\Big|}\,.$$

Since $\varphi(\vec{p})$ is arbitrary, we conclude from the last formula that under Lorentz transformations the the delta-function of three-momentum transforms as

$$\delta(\vec{p}^{\,\prime} - \vec{q}^{\,\prime}) = \frac{\delta(\vec{p} - \vec{q})}{\Big|\det\Big(\Lambda^i_j - \Lambda^i_0 \frac{p_j}{p^0}\Big)\Big|}\,. \tag{3.28}$$

Denote $A = ||\Lambda^j_i||$ and introduce two three-vectors $\vec{u}$ and $\vec{v}$ with components Λ^i_0 and $-\frac{p_i}{p_0}$, respectively. Then $\det\Big(\Lambda^i_j - \Lambda^i_0 \frac{p_j}{p^0}\Big) = \det(A + u \otimes v^t)$. This determinant can be computed by using the matrix determinant lemma[7]

$$\det(A + u \otimes v^t) = (1 + v^t A^{-1} u)\det A\,.$$

Therefore, we get

$$\det\Big(\Lambda^i_j - \Lambda^i_0 \frac{p_j}{p^0}\Big) = \Big(1 - \frac{p_i}{p_0}(A^{-1})^i_{\ j}\Lambda^j_0\Big)\det A\,,$$

where the matrix elements of A are identified with Λ^j_i. The relation $\eta^{\mu\nu}\Lambda^\alpha_\mu \Lambda^\beta_\nu = \eta^{\alpha\beta}$ implies

$$\Lambda^0_0\Lambda^j_0 - \Lambda^0_k\Lambda^j_k = 0\,,$$

which gives in its turn

$$(A^{-1})^i_{\ j}\Lambda^j_0 = \frac{1}{\Lambda^0_0}\Lambda^0_k (A^{-1})^i_{\ j}\Lambda^j_k = \frac{\Lambda^0_i}{\Lambda^0_0}\,.$$

Thus,

$$\det\Big(\Lambda^i_j - \Lambda^i_0 \frac{p_j}{p^0}\Big) = \Big(1 - \frac{p_i}{p_0}\frac{\Lambda^0_i}{\Lambda^0_0}\Big)\det A = \Big(\Lambda^0_0\, p^0 + \Lambda^0_i p^i\Big)\frac{\det(A)}{p_0\Lambda^0_0} = \frac{p^{0^\prime}}{p^0}\frac{\det A}{\Lambda^0_0}\,,$$

where $p^{0^\prime}$ is the Lorentz transformed p^0 component $p^{0^\prime} = \Lambda^0_0\, p^0 + \Lambda^0_i p^i$. Further, from $\eta^{00}\Lambda^i_0\Lambda^j_0 + \eta^{kk}\Lambda^i_k\Lambda^j_k = \eta^{ij}$ we infer $(AA^t)^j_{\ i} = \delta^j_i + \Lambda^i_0\Lambda^j_0$ and using the matrix determinant lemma once again, we obtain $(\det A)^2 = 1 + \Lambda^i_0\Lambda^i_0$. Finally, from $\eta_{00}\Lambda^0_0\Lambda^0_0 + \eta_{ii}\Lambda^i_0\Lambda^i_0 = \eta_0 0$, we get $\Lambda^0_0\Lambda^0_0 - \Lambda^i_0\Lambda^i_0 = 1$ and, therefore, $(\det A)^2 = 1 + \Lambda^i_0\Lambda^i_0 = (\Lambda^0_0)^2$, which implies $\det A = \pm\Lambda^0_0$. In this way we have shown that

$$p^{0^\prime}\,\delta(\vec{p}^{\,\prime} - \vec{q}^{\,\prime}) = p^0 \delta(\vec{p} - \vec{q})\,.$$

As a result $\langle \Lambda\vec{p}|\Lambda\vec{q}\rangle = \langle \vec{p}|\vec{q}\rangle$.

[7] In this relation, see also the Sherman-Morrison formula, which computes $(A + u \otimes v^t)^{-1}$.

Hamiltonian and symmetries

In the quantum theory of the free Klein-Gordon field we define the Hamiltonian as the following operator

$$H = \int \mathrm{d}\vec{p}\,\omega(\vec{p})a^\dagger(\vec{p})a(\vec{p})\,.$$

Here we deliberately put the creation operator on the left from the annihilation one. Such ordering of creation and annihilation operators, called the normal ordering, makes the energy of any Fock state well-defined. Application of H to an arbitrary state containing n particles with momenta $\vec{p}_1,\ldots,\vec{p}_n$, shows that this state is an eigenstate of H with the eigenvalue being simply the sum of individual energies of the corresponding particles

$$H|\vec{p}_1\ \ldots\vec{p}_n\rangle = \sum_{i=1}^{n}\hbar\omega(\vec{p}_i)|\vec{p}_1\ \ldots\vec{p}_n\rangle\,.$$

In general, for a set of fields $\phi_1(x_1),\phi_2(x_2),\ldots,\phi_k(x_k)$ the *normal-ordered product*

$$:\phi_1(x_1)\phi_2(x_2)\ldots\phi_k(x_k):$$

is defined as the usual product but with all creation operators being on the left from all annihilation operators.

Let us again return to the energetic-type expansion of $\phi(x)$

$$\phi(x) = c\int\frac{\mathrm{d}\vec{p}}{(2\pi\hbar)^{3/2}}\,\frac{1}{\sqrt{2\omega(\vec{p})}}\Big[a^*(\vec{p})\,e^{\frac{i}{\hbar}(Et-\vec{p}\vec{x})}+a(\vec{p})\,e^{-\frac{i}{\hbar}(Et-\vec{p}\vec{x})}\Big]\,.$$

The field operators which are positive and negative frequency parts of $\phi(x)$ read

$$\begin{aligned}\phi^+(\vec{x},0) &= c\int\frac{\mathrm{d}\vec{p}}{(2\pi\hbar)^{3/2}}\,\frac{1}{\sqrt{2\omega(\vec{p})}}a^\dagger(\vec{p})\,e^{-\frac{i}{\hbar}\vec{p}\vec{x}}\,, \qquad (3.29)\\ \phi^-(\vec{x},0) &= c\int\frac{\mathrm{d}\vec{p}}{(2\pi\hbar)^{3/2}}\,\frac{1}{\sqrt{2\omega(\vec{p})}}a(\vec{p})\,e^{\frac{i}{\hbar}\vec{p}\vec{x}}\,.\end{aligned}$$

We have

$$\phi^+(\vec{x},0)|0\rangle = \frac{c}{(2\pi\hbar)^{3/2}}\int\frac{\mathrm{d}\vec{q}}{2\omega(\vec{q})}\,e^{-\frac{i}{\hbar}\vec{q}\vec{x}}|\vec{q}\rangle\,.$$

If we take the inner product of this state with the momentum eigenstate $|\vec{p}\rangle$, we get

$$\langle\vec{p}|\frac{\phi^+(\vec{x},0)}{\hbar c}|0\rangle = \frac{1}{(2\pi\hbar)^{3/2}}\int\frac{\mathrm{d}\vec{q}}{2\hbar\omega(\vec{q})}\,e^{-\frac{i}{\hbar}\vec{q}\vec{x}}\underbrace{\langle\vec{p}|\vec{q}\rangle}_{2\hbar\omega(p)\delta(\vec{p}-\vec{q})} = \frac{e^{-\frac{i}{\hbar}\vec{p}\vec{x}}}{(2\pi\hbar)^{3/2}} = \langle\vec{p}|\vec{x}\rangle\,,$$

where in the last relation we recall the scalar product between momentum and position eigenstates.[8] Thus, we see that we can interpret $\phi^+(\vec{x},0)$ as an operator which creates out of the vacuum a particle at position $\vec{x}$.

3.3 Commutation and Green's functions

In the theory of interacting fields solutions of inhomogeneous field equations with point-like sources play a prenominal role. They are known as Green's functions and they include, in particular, the

[8] Normalization of the wave function of a free particle is $\int\mathrm{d}^3x\,\overline{\langle\vec{q}|\vec{x}\rangle}\langle\vec{p}|\vec{x}\rangle = \delta(\vec{p}-\vec{q})$, which is standard in quantum mechanics.

retarded and advanced Green functions, and also the so-called causal Green's function, the latter is also known as the Feynmann propagator. Below we discuss these functions in the context of the Klein-Gordon field. We start, however, from solutions of the Klein-Gordon equation without sources, which are related to commutators of quantum fields at different space-time points and, therefore, are known as commutation functions.

Pauli-Jordan function

Given the commutation relations between creation and annihilation operators, we are now in a position to compute the commutator of quantum fields at different space and time points. We define[9]

$$\big[\phi(\vec{x},t),\phi(\vec{x}',t')\big] \quad = \quad i\hbar c\,\mathscr{D}(x-x')\,.$$

Obviously, $\mathscr{D}(x)$ is a function rather than an operator; it is known as the Pauli-Jordan function. By construction, the Paui-Jordan a solution of the Klein-Gordon equation with respect to either x or x'. It has the following explicit expression

$$\mathscr{D}(x) = -\frac{\epsilon(t)}{2\pi}\left[\delta\big(x^2\big) - \frac{1}{2}\Big(\frac{mc}{\hbar}\Big)^2\theta\big(x^2\big)\frac{J_1\Big(\frac{mc}{\hbar}\sqrt{x^2}\Big)}{\frac{mc}{\hbar}\sqrt{x^2}}\right]. \tag{3.30}$$

Here $(x-x')^2 = c^2(t-t')^2 - (\vec{x}-\vec{x}')^2$ is the Lorentz-invariant four-interval, $\theta(x)$ is the Heaviside theta function and $J_1(x)$ is the Bessel function of the first kind. Also $\epsilon(t)$ denotes the sign function

$$\epsilon(t) = \begin{cases} +1 & t>0 \\ 0 & t=0 \\ -1 & t<0 \end{cases}$$

Thus, for $t=t'$ the equal-time commutator $\big[\phi(\vec{x},t),\phi(\vec{x}',t)\big]$ vanishes.

Importantly, from the explicit form of the Pauli-Jordan function[10], we see that the commutator of two local fields vanishes if their space-time points are separated by the space-like interval. Local quantum operators taken at points that are not causally connected do commute. This is one of the important implementations of *causality* in quantum field theory.

Expanding the Pauli-Jordan function around the light-cone, we find

$$\mathscr{D}(x) = -\frac{\epsilon(t)}{2\pi}\Big[\delta(x^2) - \frac{1}{2}\Big(\frac{mc}{\hbar}\Big)^2\theta(x^2) + \dots\Big], \tag{3.31}$$

where terms which vanish when $x^2\to 0$ have been omitted. Thus, the Pauli-Jordan function has a delta-function singularity as well as a finite-discontinuity (jump) on the light-cone.

Below we give a detailed derivation of the Pauli-Jordan function.

$$\begin{aligned}
\big[\phi(\vec{x},t),\phi(\vec{x}',t')\big] \quad &= \quad \frac{\hbar c^2}{(2\pi\hbar)^3}\int\frac{\mathrm{d}\vec{p}\mathrm{d}\vec{p}'}{\sqrt{4EE'}}\Big[[a^\dagger(\vec{p}),a(\vec{p}')]e^{\frac{i}{\hbar}(Et-E't')-\frac{i}{\hbar}(\vec{p}\vec{x}-i\vec{p}'\vec{x}')} + [a(\vec{p}),a^\dagger(\vec{p}')]e^{-\frac{i}{\hbar}(Et-E't')+\frac{i}{\hbar}(\vec{p}\vec{x}-i\vec{p}'\vec{x}')}\Big] \\
&= \quad -\frac{\hbar^2c^2}{(2\pi\hbar)^3}\int\frac{\mathrm{d}\vec{p}}{2E}\Big[e^{\frac{i}{\hbar}E(t-t')-\frac{i}{\hbar}\vec{p}(\vec{x}-\vec{x}')} - e^{-\frac{i}{\hbar}E(t-t')+\frac{i}{\hbar}\vec{p}(\vec{x}-\vec{x}')}\Big] \qquad (3.32)\\
&= \quad i\hbar c\frac{-2\hbar c}{(2\pi\hbar)^3}\int\frac{\mathrm{d}\vec{p}}{2E}e^{\frac{i}{\hbar}\vec{p}(\vec{x}-\vec{x}')}\sin\frac{E(t-t')}{\hbar}\,.
\end{aligned}$$

The Pauli-Jordan function is then

$$\mathscr{D}(x-x') \quad = \quad -\frac{2\hbar c}{(2\pi\hbar)^3}\int\frac{\mathrm{d}\vec{p}}{2E}e^{\frac{i}{\hbar}\vec{p}(\vec{x}-\vec{x}')}\sin\frac{E(t-t')}{\hbar}\,. \tag{3.33}$$

[9] We use the notation $\mathscr{D}$ for solutions of the homogenious Klein-Gordon equation and $\triangle$ for Green's functions.
[10] It is of the opposite sign to the notation of Bogolubov and Shirkov but agrees with Schweber.

Now we perform integration passing to the spherical coordinates

$$\mathscr{D}(x) = -\frac{2\hbar c}{(2\pi\hbar)^3}\int_0^\infty \frac{p^2 \mathrm{d}p}{2E}\sin\frac{Et}{\hbar}\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi \sin\theta \mathrm{d}\theta\, e^{\frac{i}{\hbar}px\cos\theta}\,.$$

Here $r=\sqrt{\vec{x}^2}$, $p=\sqrt{\vec{p}^2}$. Integrating over angles gives

$$\mathscr{D}(x) = -\frac{2\hbar c}{(2\pi\hbar)^2 r}\int_0^\infty \frac{p\,\mathrm{d}p}{E(p)}\sin\frac{pr}{\hbar}\sin\frac{E(p)t}{\hbar}\,,$$

which can be represented in the form

$$\mathscr{D}(x) = \frac{1}{4\pi^2}\frac{1}{r}\frac{\partial}{\partial r}\int_{-\infty}^\infty \frac{\mathrm{d}e}{\sqrt{e^2+m^2c^4}}\cos\frac{er}{c\hbar}\sin\frac{\sqrt{e^2+m^2c^4}t}{\hbar},$$

where we changed the variable $e=cp$ and used the fact that the integrand is an even function of p. This motivates to introduce the function

$$F(r,t) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{\mathrm{d}e}{\sqrt{e^2+m^2c^4}}\cos\frac{er}{c\hbar}\sin\frac{\sqrt{e^2+m^2c^4}t}{\hbar}\,.$$

Now to evaluate this integral we make a change of variables $e=mc^2\sinh\varphi$, where $-\infty\leqslant\varphi<\infty$. Then $\sqrt{e^2+m^2c^4}=mc^2\cosh\varphi>0$, so that

$$\begin{aligned} F(r,t) &= \frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\varphi\cos\Big(\frac{mcr}{\hbar}\sinh\varphi\Big)\sin\Big(\frac{mc^2t}{\hbar}\cosh\varphi\Big) = \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{d}\varphi\Big[\sin\Big(\frac{mcr}{\hbar}\sinh\varphi+\frac{mc^2t}{\hbar}\cosh\varphi\Big)-\sin\Big(\frac{mcr}{\hbar}\sinh\varphi-\frac{mc^2t}{\hbar}\cosh\varphi\Big)\Big]\,. \end{aligned}$$

Then we need to consider three different cases depending on the inequalities between ct and r. For definiteness, we choose $ct>r>0$ and then make the change of variables

$$\frac{ct}{\sqrt{(ct)^2-r^2}}=\cosh\varphi_0\,,\qquad \frac{r}{\sqrt{(ct)^2-r^2}}=\sinh\varphi_0\,,$$

so that

$$\begin{aligned} F(r,t) &= \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{d}\varphi\Big[\sin\Big(\frac{mc}{\hbar}\sqrt{(ct)^2-r^2}\cosh(\varphi+\varphi_0)\Big)+\sin\Big(\frac{mc}{\hbar}\sqrt{(ct)^2-r^2}\cosh(\varphi-\varphi_0)\Big)\Big] = \\ &= \frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\varphi\,\sin\Big(\frac{mc}{\hbar}\sqrt{(ct)^2-r^2}\cosh(\varphi)\Big) = J_0\Big(\frac{mc}{\hbar}\sqrt{(ct)^2-r^2}\Big)\,, \end{aligned}$$

where $J_0(x)$ is the Bessel function. Doing the other cases one finds the complete result

$$F(r,t)=\begin{cases} +J_0\Big(\frac{mc}{\hbar}\sqrt{(ct)^2-r^2}\Big) & \text{for}\quad ct>r \\ 0 & \text{for}\quad -r<ct<r \\ -J_0\Big(\frac{mc}{\hbar}\sqrt{(ct)^2-r^2}\Big) & \text{for}\quad ct<-r \end{cases}$$

This result can be written as a single formula

$$F(r,t)=\epsilon(t)\theta\Big((ct)^2-r^2\Big)J_0\Big(\frac{mc}{\hbar}\sqrt{(ct)^2-r^2}\Big)\,.$$

Thus,

$$\mathscr{D}(x)=\frac{\epsilon(t)}{4\pi r}\frac{\partial}{\partial r}\left[\theta\Big((ct)^2-r^2\Big)J_0\Big(\frac{mc}{\hbar}\sqrt{(ct)^2-r^2}\Big)\right],$$

which leads to the formula (3.30).

Retarded Green's function

Above we have introduced the important Pauli-Jordan function which equals to the commutator of two field operators at two arbitrary space-time points. Now define the following function[11]

$$\triangle_{\mathrm{ret}}(x-x') = \theta(t-t')[\phi(\vec{x},t),\phi(\vec{x}',t')] = i\hbar c\,\theta(t-t')\mathscr{D}(x-x')\,. \tag{3.34}$$

[11] The function $\mathscr{D}(x)$ is real, while $\triangle_{\mathrm{ret}}(x)$ is purely imaginary.

One comment is in order. We hope that the reader already acquinted enough experience with accounting in the formulae the fundamental constants c and $\hbar$. To proceed, it is advantageous to adopt the natural system of units $c = 1 = \hbar$. Since $[\hbar] = p \cdot \ell$, and $[p] = m \cdot c$, in this system the main remaining parameter is the mass m, while the length $\ell \sim 1/m$. The actual constants c and $\hbar$ can aways be restored on dimensional grounds. In any equation the physical (engineering) dimension of the left hand side must be equal to that of the right hand side, otherwise an equation makes no sence.

With the natural system of units at hand, we now demonstrate that $\triangle_{\rm ret}(x)$ is nothing else but the retarded Green's function for the Klein-Gordon equation. This can be done at least in two different ways. The simplest one is to act on $\triangle_{\rm ret}$ with the Klein-Gordon operator and use the fact that $\phi(\vec{x}, t)$ solves the Klein-Gordon equation. Indeed,[12]

$$\begin{aligned}
\Big(\partial_t^2 - \partial_i^2 + m^2\Big)\Big(\theta(t-t')[\phi(\vec{x},t), \phi(\vec{x}',t')]\Big) \quad &= \\
&= \partial_t(\underbrace{\delta(t-t')[\phi(\vec{x},t), \phi(\vec{x}',t')]}_{=0} + \theta(t-t')[\pi(\vec{x},t), \phi(\vec{x}',t')]) \\
&+ \theta(t-t')[(-\partial_i^2 + m^2)\phi(\vec{x},t), \phi(\vec{x}',t')] \\
&= \delta(t-t')[\pi(\vec{x},t), \phi(\vec{x}',t')] + \theta(t-t')[\underbrace{(\partial_t^2 - \partial_i^2 + m^2)\phi(\vec{x},t)}_{=0}, \phi(\vec{x}',t')] = \\
&= -i\,\delta(t-t')\delta(\vec{x}-\vec{x}') = -i\,\delta^{(4)}(x-x')\,.
\end{aligned}$$

Further, in accord with the definition, $\triangle_{\rm ret}(x)$ vanishes for $t < t'$, which is the characteristic property of the retarded Green's function.

The second way to show that $\triangle_{\rm ret}(x)$ is the Green's function is to analyze an integral representation

$$\triangle_{\rm ret}\,(x) \quad = \quad -\frac{\theta(t)}{(2\pi)^3}\int \frac{{\rm d}\vec{k}}{2k^0}\Big[e^{ik^0 t} - e^{-ik^0 t}\Big]e^{+i\vec{k}\vec{x}}\,,$$

which follows from the formula (3.32) upon taking into account that, in the natural units, $\vec{p} = \vec{k}$ and $E = k^0$, where $k^0 = \sqrt{\vec{k}^2 + m^2}$. Further, we consider the following integral (k^0 is an integration variable)

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{e^{-ik^0 t}{\rm d}k^0}{(k^0 + i\epsilon)^2 - \vec{k}^2 - m^2} = \\
&\qquad = \int_{-\infty}^{\infty} \frac{{\rm d}k^0}{2\sqrt{\vec{k}^2 + m^2}}\left[\frac{e^{-ik^0 t}}{k^0 - \sqrt{\vec{k}^2 + m^2} + i\epsilon} - \frac{e^{-ik^0 t}}{k^0 + \sqrt{\vec{k}^2 + m^2} + i\epsilon}\right].
\end{aligned}$$

To compute this integral, we note that for $t > 0$ the integration contour can be closed in the lower half k^0-plane, in which case the contour encloses two poles at

$$k^0 = \sqrt{\vec{k}^2 + m^2} - i\epsilon \quad \text{and} \quad k^0 = -\sqrt{\vec{k}^2 + m^2} - i\epsilon\,.$$

Applying Cauchy's theorem, we therefore find

$$\int_{-\infty}^{\infty} \frac{e^{-ik^0 t}{\rm d}k^0}{(k^0 + i\epsilon)^2 - \vec{k}^2 - m^2} = \frac{2\pi i}{2\sqrt{\vec{k}^2 + m^2}}\Big[e^{i\sqrt{\vec{k}^2+m^2}\,t} - e^{-i\sqrt{\vec{k}^2+m^2}\,t}\Big].$$

[12] With c and $\hbar$ restored the equation below will read $\partial_\mu\partial^\mu(\theta(t-t')[\phi(\vec{x},t), \phi(\vec{x}',t')]) = -i\hbar c\,\delta^{(4)}(x-x')$.

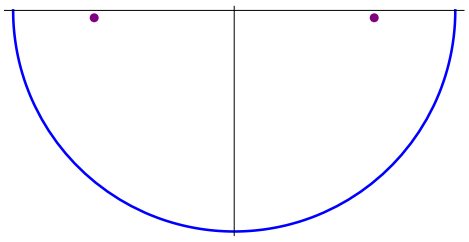

Figure 3.1: The integration contour and position of poles of the integrand for the retarded Green function. Poles lie in the lower k^0-half-plane.

For $t < 0$ the integration contour can be closed in the upper half-plane where there are no poles and, therefore, the corresponding integral is zero. Thus, we ontain the following integral representation

$$\triangle_{\rm ret}\,(x) \quad = \quad \int \frac{{\rm d}^4k}{(2\pi)^4}\,\frac{i}{(k^0+i\epsilon)^2-\vec{k}^2-m^2}e^{-ikx}\,.$$

Now we act on this expression with the Klein-Gordon operator

$$\left(\partial_\mu\partial^\mu+m^2\right)\triangle_{\rm ret}\,(x) \quad = \quad -\lim_{\epsilon\to 0} i\int \frac{{\rm d}^4k}{(2\pi)^4}\,\frac{(k^0)^2-\vec{k}^2-m^2}{(k^0+i\epsilon)^2-\vec{k}^2-m^2}e^{-ikx} = -i\,\delta^{(4)}(x)\,.$$

Thus, we conclude again that $\triangle_{\rm ret}(x)$ is the Green's function for the Klein-Gordon equation and it is non-zero only in the forward light-cone, *i.e.* by definition it is the retarded Green's function. Analogously, one can introduce the advanced Green's function and study its properties.

Feynman propagator

In quantum field theory a special role is played by the causal Green's function $\triangle(x-x')$, which we also called the Feynman propagator. This function describes a causal relationship between the processes of creation and annihilation of particles in different space-time points x and x'.

The following interpretation of the field amplitudes is natural. The process of creating first a particle at a point x' with its subsequent annihilation at x is described by the amplitude

$$\langle 0|\phi(x)\phi(x')|0\rangle = \langle 0|\phi^-(x)\phi^+(x')|0\rangle = \langle 0|[\phi^-(x)\phi^+(x')]|0\rangle = i\hbar c\,\mathscr{D}^-(x-x')\,, \tag{3.35}$$

where it is then natural to consider $t > t'$. Analogously, the same amplitude can be looked as corresponding to first destroying a particle at x and creating a new one at x', in which case it is natural to have $t < t'$. Here $\mathscr{D}^-(x-x')$ is a negative frequency part of the Pauli-Jordan function. In general, the positive and negative frequency parts of the Pauli-Jordan function are defined as

$$[\phi^\pm(\vec{x},t),\phi^\mp(\vec{x}',t')] \quad = \quad i\hbar c\,\mathscr{D}^\pm(x-x')\,.$$

Then the Feynman propagator is defined as follows

$$\begin{aligned}\triangle\,(x-x') \quad &= \quad \theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle + \theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle = \\ &= \quad \theta(t-t')\,i\hbar c\,\mathscr{D}^-(x-x') + \theta(t'-t)\,i\hbar c\,\mathscr{D}^-(x'-x)\,.\end{aligned} \tag{3.36}$$

In fact, $\triangle(x-x')$ can be concisely written by using the notion of the time-ordered product

$$\triangle\,(x-x') = \langle 0\mid T(\phi(x)\phi(x'))\mid 0\rangle\,. \tag{3.37}$$

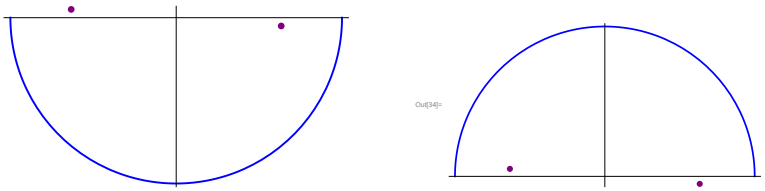

Figure 3.2: The integration contour and position of poles of the integrand for the Feynman propagator. The left figure is for $t > 0$ and the right one for $t < 0$.

Time ordering means putting operators in the order of increasing time argument from the right to the left. Colloquially speaking, the younger operator is righter it stands.[13]

For the auxiliary function $\mathscr{D}^-(x)$ it is straightforward to obtain (in the natural units)

$$\mathscr{D}^-(x) = -\frac{i}{(2\pi)^3}\int \frac{\mathrm{d}\vec{k}}{2k^0} e^{-ik^0t+i\vec{k}\vec{x}}\,.$$

Taking into account this integral representation for $\mathscr{D}^-(x)$, we arrive at

$$\triangle\,(x) = \frac{1}{(2\pi)^3}\int \mathrm{d}\vec{k}\, e^{i\vec{k}\vec{x}}\Big[\frac{\theta(t)}{2k^0}e^{-ik^0t} + \frac{\theta(-t)}{2k^0}e^{ik^0t}\Big], \qquad k^0 = \sqrt{\vec{k}^2+m^2}\,.$$

Consider now the following integral (here k^0 is an integration variable!)

$$\int \frac{\mathrm{d}k^0 e^{-ik^0t}}{k^2-m^2+i\epsilon} = \int \frac{\mathrm{d}k^0 e^{-ik^0t}}{(k^0)^2-(\vec{k}^2+m^2)+i\epsilon} \tag{3.38}$$

As the function of k^0 the integrand has poles in the complex plane at

$$k^0 = \pm\sqrt{\vec{k}^2+m^2-i\epsilon} \simeq \pm\sqrt{\vec{k}^2+m^2} \mp i\epsilon\,.$$

Expanding the integrant into simple fractions, we get

$$\int \frac{\mathrm{d}k^0 e^{-ik^0t}}{k^2-m^2+i\epsilon} = \int \frac{\mathrm{d}k^0}{2\sqrt{\vec{k}^2+m^2}}\Big[\underbrace{\frac{e^{-ik^0t}}{k^0-\sqrt{k^2+m^2}+i\epsilon}}_{t>0} - \underbrace{\frac{e^{-ik^0t}}{k^0+\sqrt{k^2+m^2}-i\epsilon}}_{t<0}\Big] =$$
$$= 2\pi i\frac{1}{2\sqrt{\vec{k}^2+m^2}}\Big[-\theta(t)e^{-i\sqrt{\vec{k}^2+m^2}t} - \theta(-t)e^{i\sqrt{\vec{k}^2+m^2}t}\Big],$$

where in the last step Cauchy's residue theorem was applied.[14] Thus, we find for $\triangle$ the following quite compact integral representation in the form of the four-fold integral

$$\triangle\,(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ikx}\,. \tag{3.39}$$

Obviously, $\triangle(x)$ is also a Green's function for the Klein-Gordon equation.

It is interesting to know how $\triangle(x)$ behaves as a function of x. Below we provide an answer to this question.

[13] This rule is easy to remember by envoking a slogan "The youth is always right!".

[14] Pay attention to the orientation of the integration path!

First, we note that $\triangle(x)$ is symmetric under the change $t \to -t$, so we can assume without loss of generality that $t > 0$. In this case the expression for $\triangle$ boils down to

$$\triangle\,(x) \,=\, \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}\vec{k}}{2\sqrt{\vec{k}^2+m^2}}\, e^{i\vec{k}\vec{x}} e^{-i\sqrt{\vec{k}^2+m^2}\,t}\,, \qquad t>0\,.$$

Switching to the spherical coordinates and integrating over angles, precisely in the same way as it was done for the Pauli-Jordan function, we obtain

$$\triangle\,(x) \,=\, \frac{1}{(2\pi)^2 r}\int_0^\infty \frac{k\mathrm{d}k\,\sin(kr)}{\sqrt{k^2+m^2}}\; e^{-i\sqrt{k^2+m^2}\,t} \,=\, -\frac{1}{8\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty}\frac{\mathrm{d}k\,\cos(kr)}{\sqrt{k^2+m^2}}\, e^{-i\sqrt{k^2+m^2}\,t}\,.$$

As above, here the notations $k=\sqrt{\vec{k}^2}$ and $r=\sqrt{\vec{x}^2}$ are used. The last formula can be rewritten as

$$\triangle\,(x) \,=\, -\frac{1}{16\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{\sqrt{k^2+m^2}}\Big(e^{i(kr-\sqrt{k^2+m^2}\,t)} + e^{-i(kr+\sqrt{k^2+m^2}\,t)}\Big)$$

Finally, introducing $k = m\sinh\varphi$, we arrive at

$$\triangle\,(x) \,=\, -\frac{1}{16\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty}\mathrm{d}\varphi\,\Big(e^{im(r\sinh\varphi - t\cosh\varphi)} + e^{-im(r\sinh\varphi + t\cosh\varphi)}\Big)\,.$$

Now we have to distinguish two cases.

1) $r > t$, that is the space-time interval is space-like $x^2 < 0$. In this case we introduce the variable φ_0 as

$$\frac{r}{\sqrt{r^2-t^2}} = \cosh\varphi_0\,, \qquad \frac{t}{\sqrt{r^2-t^2}} = \sinh\varphi_0\,.$$

$$\begin{aligned}\triangle\,(x) \;&=\; -\frac{1}{16\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty}\mathrm{d}\varphi\,\Big(e^{im\sqrt{r^2-t^2}\,\sinh(\varphi-\varphi_0)} + e^{-im\sqrt{r^2-t^2}\,\sinh(\varphi+\varphi_0)}\Big) = \\ &=\; -\frac{1}{8\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty}\mathrm{d}\varphi\, e^{im\sqrt{r^2-t^2}\,\sinh\varphi} = -\frac{1}{4\pi^2 r}\frac{\partial}{\partial r}\Big[\theta(-x^2)\,K_0(m\sqrt{-x^2})\Big]\,.\end{aligned}$$

Thus, we get

$$\triangle(x) \,=\, \frac{m}{4\pi^2\sqrt{-x^2}}\,K_1(m\sqrt{-x^2})\,, \qquad x^2<0\,,$$

where we keep for the moment a regular term only. There is also a singular term arising from differentiating $\theta(-x^2)$ over r. We consider it later.

2) $r < t$, that is the space-time interval is time-like $x^2 > 0$. In this case we introduce the variable φ_0 as follows

$$\frac{r}{\sqrt{t^2-r^2}} = \sinh\varphi_0\,, \qquad \frac{t}{\sqrt{r^2-r^2}} = \cosh\varphi_0$$

and get

$$\begin{aligned}\triangle\,(x) \;&=\; -\frac{1}{16\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty}\mathrm{d}\varphi\,\Big(e^{-im\sqrt{t^2-r^2}\,\cosh(\varphi-\varphi_0)} + e^{-im\sqrt{t^2-r^2}\,\cosh(\varphi+\varphi_0)}\Big) = \\ &= -\frac{1}{8\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty}\mathrm{d}\varphi\, e^{-im\sqrt{t^2-r^2}\,\cosh\varphi} = \frac{i}{8\pi r}\frac{\partial}{\partial r}\Big[\theta(x^2)\,H_0^{(2)}(m\sqrt{t^2-r^2})\Big]\,.\end{aligned}$$

Performing differentiation, we find the following regular term

$$\triangle\,(x) \;=\; \frac{im}{8\pi\sqrt{x^2}}\,H_1^{(2)}(m\sqrt{x^2})\,, \qquad x^2>0\,.$$

There is also a singular term, but we treat it together with the one coming from the first case.

3) Now we treat singular terms. We can combine them as follows

$$\begin{aligned}\triangle_{\mathrm{sing}}\,(x) \;&=\; \frac{i}{8\pi r}\frac{\partial\theta(x^2)}{\partial r}\,H_0^{(2)}(m\sqrt{x^2}) - \frac{1}{4\pi^2 r}\frac{\partial\theta(-x^2)}{\partial r}\,K_0(m\sqrt{-x^2}) = \\ &=\; \delta(x^2)\lim_{s\to+0}\Big[-\frac{i}{4\pi}H_0^{(2)}(ms) - \frac{1}{2\pi^2}K_0(ms)\Big] = -\frac{i}{4\pi}\delta(x^2)\,.\end{aligned}$$

Combining these results and restoring the physical units, we obtain the following space-time expression for the Feynman propagator

$$\triangle(x) = \hbar c \left[-\frac{i}{4\pi}\delta(x^2) + \theta(x^2)\frac{i}{8\pi}\Big(\frac{mc}{\hbar}\Big)^2 \frac{H_1^{(2)}\big(\frac{mc}{\hbar}\sqrt{x^2}\big)}{\frac{mc}{\hbar}\sqrt{x^2}} + \theta(-x^2)\frac{1}{4\pi^2}\Big(\frac{mc}{\hbar}\Big)^2 \frac{K_1\big(\frac{mc}{\hbar}\sqrt{-x^2}\big)}{\frac{mc}{\hbar}\sqrt{-x^2}} \right].$$

In fact, the term corresponding to $x^2 > 0$ can be obtained from the term with $x^2 < 0$ by analytic continuation. If we take $x^2 > 0$ and define[15] $\sqrt{-x^2} = i\sqrt{x^2}$, then

$$\frac{K_1\big(\frac{mc}{\hbar}\sqrt{-x^2}\big)}{\frac{mc}{\hbar}\sqrt{-x^2}} = \frac{K_1\big(\frac{mc}{\hbar}i\sqrt{x^2}\big)}{\frac{mc}{\hbar}i\sqrt{x^2}} = \frac{i\pi}{2}\frac{H_1^{(2)}\big(\frac{mc}{\hbar}\sqrt{x^2}\big)}{\frac{mc}{\hbar}\sqrt{x^2}}\,.$$

This allows one to write the following compact expression

$$\triangle(x) = \frac{\hbar c}{4\pi^2}\Big(\frac{mc}{\hbar}\Big)^2 \frac{K_1\big(\frac{mc}{\hbar}\sqrt{-x^2+i\epsilon}\big)}{\frac{mc}{\hbar}\sqrt{-x^2+i\epsilon}}$$

valid for all x. As is clear from the definition (3.37) or from the expression above, the physical dimension of the Feynman propagator is $\mathscr{E}/\ell$.

3.4 Wick's theorem – operatorial approach

Here we demonstrate how to compute the correlation functions

$$\langle 0 \mid T(\phi(x_1)\phi(x_2)\dots\phi(x_n)) \mid 0\rangle \tag{3.40}$$

by using the field commutation relations. We start from $\langle 0 \mid T(\phi(x_1)\phi(x_2) \mid 0\rangle$. We know how to compute this quantity. We have

$$\phi(x) = \phi^+(x) + \phi^-(x)\,,$$

where $\phi^+(x)$ and $\phi^-(x)$ are positive and negative frequency parts of the Klein-Gordon field.

$$\phi^+(x) \;=\; \frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}\vec{k}}{\sqrt{2k^0}}\Big[a^\dagger(\vec{k})\,e^{ik^0t - i\vec{k}\vec{x}}\Big], \qquad \phi^-(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}\vec{k}}{\sqrt{2k^0}}\Big[a(\vec{k})\,e^{-ik^0t + i\vec{k}\vec{x}}\Big].$$

Here $k^0 = \sqrt{\vec{k}^2 + m^2} > 0$. We recall that the notion of positive and negative frequency parts has been introduced in section 3.1 As a general rule, a positive-frequency solution has as its coefficient the creation operator that creates a particle, while a negative-frequency solution has as its coefficient the annihilation operator that destroyes a particle. Thus,

$$\phi^-(x)|0\rangle = 0\,, \qquad \langle 0|\phi^+(x) = 0\,.$$

Consider, for instance, $x^0 > y^0$. Then

$$\begin{aligned} T(\phi(x)\phi(y)) &= \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) = \\ &= \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^+(y)\phi^-(x) + \phi^-(x)\phi^-(y) + [\phi^-(x), \phi^+(y)]\,. \end{aligned}$$

The right hand side of the last expression is brought to the normal order – all annihilation operators are on the right of all creation operators. We can rewrite the last expression as

$$\begin{aligned} T(\phi(x)\phi(y)) &= :\phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^+(y)\phi^-(x) + \phi^-(x)\phi^-(y) : + [\phi^-(x), \phi^+(y)] = \\ &= :\phi(x)\phi(y): + [\phi^-(x), \phi^+(y)]\,. \end{aligned}$$

[15] This corresponds picking up the branch corresponding to $\sqrt{-x^2+i\epsilon}$ for $x^2 > 0$.

Now, consider $y^0 > x^0$. We get

$$\begin{aligned} T(\phi(x)\phi(y)) &= \phi^+(y)\phi^+(x) + \phi^+(y)\phi^-(x) + \phi^-(y)\phi^+(x) + \phi^-(y)\phi^-(x) = \\ &= \phi^+(y)\phi^+(x) + \phi^+(y)\phi^-(x) + \phi^+(x)\phi^-(y) + \phi^-(y)\phi^-(x) + [\phi^-(y), \phi^+(x)] = \\ &= :\phi(x)\phi(y): + [\phi^-(y), \phi^+(x)]\,. \end{aligned}$$

Let us define the contraction of two fields as follows

$$\overbrace{\phi(x)\phi(y)} = \begin{cases} \big[\phi^-(x), \phi^+(y)\big], & x^0 > y^0\,, \\ \\ \big[\phi^-(y), \phi^+(x)\big], & y^0 > x^0\,. \end{cases}$$

One can easily recognize that the contraction is nothing else but the Feynman propagator

$$\overbrace{\phi(x)\phi(y)} = \triangle(x-y)\,. \tag{3.41}$$

Thus, the relation between time-ordering and normal-ordering is given by

$$T(\phi(x)\phi(y)) =: \phi(x)\phi(y) + \overbrace{\phi(x)\phi(y)}: \ .$$

With the new notation the result above is easy to generalize to an arbitrary number of fields

$$T(\phi(x_1)\phi(x_2)\ldots\phi(x_n)) =: \phi(x_1)\phi(x_2)\ldots\phi(x_n) + \text{all possible contractions}: \ .$$

This identity is known as Wick's theorem. For instance, for the case of four fields Wick's theorem gives

$$\begin{aligned} T(\phi_1\phi_2\phi_3\phi_3) &= :\phi_1\phi_2\phi_3\phi_4 + \overbrace{\phi_1}\phi_2\phi_3\phi_4 + \overbrace{\phi_1\phi_2}\phi_3\phi_4 + \overbrace{\phi_1\phi_2\phi_3\phi_4} + \\ &+ \phi_1\overbrace{\phi_2}\phi_3\phi_4 + \phi_1\overbrace{\phi_2\phi_3}\phi_4 + \phi_1\phi_2\overbrace{\phi_3\phi_4} + \\ &+ \overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4} + \overbrace{\phi_1\overbrace{\phi_2}\phi_3\phi_4} + \overbrace{\phi_1\overbrace{\phi_2}\phi_3\phi_4}: \ . \end{aligned}$$

Any contraction can be replaced by the corresponding Feynman propagator which can be taken out of the sign of the normal product, for instance,

$$\phi_1\overbrace{\phi_2\phi_3}\phi_4 = \triangle(x_2 - x_4) : \phi_1\phi_3 : \ .$$

In the vacuum expectation value (3.40), any term in which there remain uncontructed operators gives zero. Only fully contracted terms survive, which for our example means that

$$\langle 0 \mid T(\phi_1\phi_2\phi_3\phi_3) \mid 0\rangle = \triangle(x_1 - x_2)\triangle(x_3 - x_4) + \triangle(x_1 - x_3)\triangle(x_2 - x_4) + \triangle(x_1 - x_4)\triangle(x_2 - x_3)\,.$$

Thus, we found the same formula as by using the path integral approach.

The simplest way to prove Wick's theorem is to use the mathematical induction. Assume that the theorem is true for $m-1$ fields and try to prove it for m fields. Without loss of generality, we can restrict ourselves to the case $x_1^0 \geqslant x_2^0 \geqslant x_3^0 \ldots \geqslant x_m^0$. Then we have

$$T(\phi_1\phi_2\ldots\phi_m) = \phi_1\phi_2\ldots\phi_m = \phi_1 T(\phi_2\ldots\phi_m)\,.$$

Now we apply Wick's theorem to $m-1$ fields

$$T(\phi_1\phi_2\ldots\phi_m) = (\phi_1^+ + \phi_1^-)\left\{:\phi_2\ldots\phi_m + \begin{array}{l}\text{all possible contractions} \\ \text{not involving } \phi_1\end{array} :\right\}\,. \tag{3.42}$$

Here ϕ_1^+ can be moved in inside the normal product as it depends on the creation operator only. As to ϕ_1^- we have

$$\begin{aligned}\phi_1^-:\phi_2\dots\phi_m: &= :\phi_2\dots\phi_m:\phi_1^- + \big[\phi_1^-,:\phi_2\dots\phi_m:\big] = \\ &= :\phi_1^-\phi_2\dots\phi_m: + :\big[\phi_1^-,\phi_2^+\big]\phi_3\dots\phi_m: + \dots + \phi_2\dots\big[\phi_1^-,\phi_m^+\big] := \\ &= :\phi_1^-\phi_2\dots\phi_m + \overbrace{\phi_1\phi_2}\phi_3\dots\phi_m + \overbrace{\phi_1\phi_2\phi_3}\dots\phi_m + \dots :\end{aligned}$$

Similarly, a term in (3.42) involving one contraction will produce all possible terms involving a single contraction of ϕ_1 with one of the other fields. Doing this with all the terms we eventually get all possible contractions of all the fields, including ϕ_1. This completes the induction.

3.5 Appendices

3.5.1 Fock space formalism in quantum mechanics

The importance of the formalism of second quantization derives from the fact that it permits performing calculations which automatically take into account the combinatorial aspects arising from the particle statistics (Bose-Einstein or Fermi-Dirac). In this appendix we recall the Fock space formalism in the non-relativistic quantum mechanics.

Suppose we have a quantum-mechanical system of n identical *bosonic* particles described by the Schrödinger operator

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{n}\vec{\nabla}_i^2 + \sum_{i<j}V(\vec{x}_i,\vec{x}_j)\,, \tag{3.43}$$

where we assumed for definiteness that particles interact by means of a pairwise potential $V(\vec{x}_i,\vec{x}_j)$. The system is described by the wave function $\psi(\vec{x}_1,\dots,\vec{x}_n)$ whose time development is governed by the Schrödinger equation

$$\Big[-\frac{\hbar^2}{2m}\sum_{i=1}^{n}\vec{\nabla}_i^2 + \sum_{i<j}V(\vec{x}_i,\vec{x}_j)\Big]\psi(\vec{x}_1,\dots,\vec{x}_n,t) = i\hbar\frac{\partial}{\partial t}\psi(\vec{x}_1,\dots,\vec{x}_n,t)\,. \tag{3.44}$$

This gives a particular realization of our quantum-mechanical system, as we have chosen to realize it in a particular Hilbert space being the space of square-integrable functions totally symmetric in n coordinate variables $\vec{x}_i$. Another realization would be, for instance, to take a multi-particle wave function in the momentum representation. Yet, we introduce here a novel Hilbert space, called the Fock space, and provide the corresponding realization of our system in this space.

Introduce abstract operators $\Phi^\dagger(\vec{x})$ and $\Phi(\vec{x})$ which satisfy the commutation relations

$$\begin{aligned}&\big[\Phi(\vec{x}_1),\Phi^\dagger(\vec{x}_2)\big] = \delta(\vec{x}_1-\vec{x}_2)\,,\\ &\big[\Phi(\vec{x}_1),\Phi(\vec{x}_2)\big] = 0\,,\quad \big[\Phi^\dagger(\vec{x}_1),\Phi^\dagger(\vec{x}_2)\big] = 0\,.\end{aligned} \tag{3.45}$$

These algebraic relations are the same as those of creation and annihilation operators. We therefore choose the vacuum state $|0\rangle$ such that it is annihilated by $\Phi(\vec{x})$: $\Phi(\vec{x})|0\rangle = 0$. States in this space, known as the Fock space, are obtained by applying to the vacuum any number of creation operators $\Phi^\dagger$. For instance, a state which contains n particles is given by a superposition

$$|\chi\rangle_n = \frac{1}{\sqrt{n!}}\int[\mathrm{d}x]\,\chi_n(\vec{x}_1,\dots\vec{x}_n,t)\,\Phi^\dagger(\vec{x}_1)\Phi^\dagger(\vec{x}_2)\dots\Phi^\dagger(\vec{x}_n)|0\rangle\,.$$

Here $\chi_n(\vec{x}_1,\dots\vec{x}_n,t)$ is the coefficient function, $[\mathrm{d}x] = \prod_{i=1}^n \mathrm{d}\vec{x}_i$ and we assume that the time-dependence of $|\chi\rangle_n$ enters through this function. Since all $\Phi^\dagger(\vec{x}_i)$ entering the definition of the state commute with each other, the function $\chi(\vec{x}_1,\dots\vec{x}_n)$ is automatically symmetric under interchange of coordinates. The scalar product in this Hilbert space is defined as follows. First we define the conjugate vacuum $\langle 0| \equiv |0\rangle^\dagger$ which satisfies the relations

$$\langle 0|\Phi^\dagger(\vec{x}) = 0\,, \qquad \langle 0|0\rangle = 1\,.$$

Then for any two vectors corresponding to particle numbers n and m we have

$$\begin{aligned} {}_m\langle\zeta|\chi\rangle_n &= \frac{1}{\sqrt{m!n!}}\int [\mathrm{d}y][\mathrm{d}x]\,\zeta^*_m(\vec{y}_1,\dots,\vec{y}_m)\chi_n(\vec{x}_1,\dots,\vec{x}_n)\,\times \\ &\times\langle 0|\Phi(\vec{y}_m)\dots\Phi(\vec{y}_1)\Phi^\dagger(\vec{x}_1)\dots\Phi^\dagger(\vec{x}_n)|0\rangle = \delta_{mn}\int [\mathrm{d}x]\,\zeta^*_n(\vec{x}_1,\dots,\vec{x}_n)\chi_n(\vec{x}_1,\dots,\vec{x}_n). \end{aligned}$$

In particular, for the square of the norm of a state with an arbitrary number of particles we have

$$\langle\chi|\chi\rangle = \sum_{n=0}^{\infty}\int [\mathrm{d}x]\,|\chi(\vec{x}_1,\dots,\vec{x}_n)|^2\,.$$

Consider the following operator acting in the Fock space

$$\mathrm{H} = -\frac{\hbar^2}{2m}\int \mathrm{d}\vec{x}\,\Phi^\dagger(\vec{x})\vec{\nabla}^2\Phi(\vec{x}) + \frac{1}{2}\int \mathrm{d}\vec{x}\mathrm{d}\vec{y}\;\Phi^\dagger(\vec{x})\Phi^\dagger(\vec{y})V(\vec{x},\vec{y})\Phi(\vec{x})\Phi(\vec{y})\,. \tag{3.46}$$

Note that the commutation relations (3.45) imply that the physical dimension $[\Phi]$ of the operator Φ is $[\Phi] = \ell^{-3/2}$, where ℓ is a unit of length. The operator H is called the *second quantized Hamiltonian*. Integrating by parts in the first term we can also write it in the form

$$\mathrm{H} = \frac{\hbar^2}{2m}\int \mathrm{d}\vec{x}\,\partial_i\Phi^\dagger(\vec{x})\partial_i\Phi(\vec{x}) + \frac{1}{2}\int \mathrm{d}\vec{x}\mathrm{d}\vec{y}\;\Phi^\dagger(\vec{x})\Phi^\dagger(\vec{y})V(\vec{x},\vec{y})\Phi(\vec{x})\Phi(\vec{y})\,. \tag{3.47}$$

Now we will show that the equation

$$\mathrm{H}|\chi(t)\rangle_n = i\partial_t|\chi(t)\rangle_n \tag{3.48}$$

is equivalent to (3.44) upon identifying $\chi_n(\vec{x}_1,\dots,\vec{x}_n)$ with $\psi(\vec{x}_1,\dots,\vec{x}_n)$. We apply

$$\mathrm{H}|\chi\rangle_n = \frac{1}{\sqrt{n!}}\int [\mathrm{d}x]\,\chi_n(\vec{x}_1,\dots\vec{x}_n,t)\,\mathrm{H}\,\Phi^\dagger(\vec{x}_1)\Phi^\dagger(\vec{x}_2)\dots\Phi^\dagger(\vec{x}_n)|0\rangle\,.$$

Further computation of how the second quantized Hamiltonian H acts on the state

$$\Phi^\dagger(\vec{x}_1)\Phi^\dagger(\vec{x}_2)\dots\Phi^\dagger(\vec{x}_n)|0\rangle$$

will be split in two parts. First, we compute the action of the kinetic part of (3.46) on the n-particle state and, second, we compute the action of the potential part. The whole computation is a bit tedious but it is worth doing it in detail. We have

$$\frac{1}{\sqrt{n!}}\int[\mathrm{d}x]\int \mathrm{d}\vec{x}\,\Big(-\frac{\hbar^2}{2m}\Big)\chi_n(\vec{x}_1,\dots\vec{x}_n,t)\,\Phi^\dagger(\vec{x})\vec{\nabla}^2\Phi(\vec{x})\,\Phi^\dagger(\vec{x}_2)\dots\Phi^\dagger(\vec{x}_n)|0\rangle\,=$$

$$=\frac{1}{\sqrt{n!}}\int[\mathrm{d}x]\mathrm{d}\vec{x}\sum_{i=1}^{n}\Big(-\frac{\hbar^2}{2m}\Big)\chi_n(\vec{x}_1,\dots\vec{x}_n,t)\,\Phi^\dagger(\vec{x})\Phi^\dagger(x_1)\dots\vec{\nabla}^2_x[\Phi(\vec{x}),\Phi^\dagger(\vec{x}_i)]\dots\Phi^\dagger(\vec{x}_n)|0\rangle=$$

$$=\frac{1}{\sqrt{n!}}\int[\mathrm{d}x]\mathrm{d}\vec{x}\sum_{i=1}^{n}\Big(-\frac{\hbar^2}{2m}\Big)\chi_n(\vec{x}_1,\dots\vec{x}_n,t)\,\Phi^\dagger(\vec{x})\Phi^\dagger(\vec{x}_1)\dots\vec{\nabla}^2_x\delta(\vec{x}-\vec{x}_i)\dots\Phi^\dagger(\vec{x}_n)|0\rangle=$$

$$=\frac{1}{\sqrt{n!}}\int[\mathrm{d}x]\mathrm{d}\vec{x}\sum_{i=1}^{n}\Big(-\frac{\hbar^2}{2m}\Big)\chi_n(\vec{x}_1,\dots\vec{x}_n,t)\,\Phi^\dagger(\vec{x})\Phi^\dagger(\vec{x}_1)\dots\vec{\nabla}^2_{x_i}\delta(\vec{x}-\vec{x}_i)\dots\Phi^\dagger(\vec{x}_n)|0\rangle=$$

$$=\frac{1}{\sqrt{n!}}\int[\mathrm{d}x]\mathrm{d}\vec{x}\sum_{i=1}^{n}\Big(-\frac{\hbar^2}{2m}\vec{\nabla}^2_{x_i}\chi_n(\vec{x}_1,\dots\vec{x}_n,t)\Big)\,\Phi^\dagger(\vec{x})\Phi^\dagger(\vec{x}_1)\dots\delta(\vec{x}-\vec{x}_i)\dots\Phi^\dagger(\vec{x}_n)|0\rangle=$$

$$=\frac{1}{\sqrt{n!}}\int[\mathrm{d}x]\sum_{i=1}^{n}\Big(-\frac{\hbar^2}{2m}\vec{\nabla}^2_{x_i}\chi_n(\vec{x}_1,\dots\vec{x}_n,t)\Big)\,\Phi^\dagger(\vec{x}_1)\Phi^\dagger(\vec{x}_2)\dots\dots\Phi^\dagger(\vec{x}_n)|0\rangle\,.$$

Here we have integrated $\vec{\nabla}^2_{x_i}$ by parts with the subsequent integration over x and also used the fact that the operators $\Phi^\dagger(\vec{x}_i)$ commute between themselves, which allowed us to bring them to the original order. Then, for the potential part we get

$$\frac{1}{2}\frac{1}{\sqrt{n!}}\int[\mathrm{d}x]\mathrm{d}\vec{x}\mathrm{d}\vec{y}\,\chi_n(\vec{x}_1,\dots\vec{x}_n,t)\,\Phi^\dagger(\vec{x})\Phi^\dagger(\vec{y})V(\vec{x},\vec{y})\Phi(\vec{x})\Phi(\vec{y})\,\Phi^\dagger(\vec{x}_1)\dots\Phi^\dagger(\vec{x}_n)|0\rangle\,=$$

$$=\frac{1}{2}\frac{1}{\sqrt{n!}}\int[\mathrm{d}x]\mathrm{d}\vec{x}\mathrm{d}\vec{y}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\chi_n(\vec{x}_1,\dots\vec{x}_n,t)\,\Phi^\dagger(\vec{x})\Phi^\dagger(\vec{y})V(\vec{x},\vec{y})\,\times$$

$$\times\,\Phi^\dagger(\vec{x}_1)\dots\underbrace{[\Phi(\vec{y}),\Phi^\dagger(\vec{x}_j)]}_{\delta(\vec{y}-\vec{x}_j)}\dots\underbrace{[\Phi(\vec{x}),\Phi^\dagger(\vec{x}_i)]}_{\delta(\vec{x}-\vec{x}_i)}\dots\Phi^\dagger(\vec{x}_n)|0\rangle=$$

$$=\frac{1}{2}\frac{1}{\sqrt{n!}}\int[\mathrm{d}x]\sum_{i=1}^{n}\sum_{j\neq i}^{n}V(\vec{x}_i,\vec{x}_j)\chi_n(\vec{x}_1,\dots\vec{x}_n,t)\,\Phi^\dagger(\vec{x}_1)\Phi^\dagger(\vec{x}_2)\dots\dots\Phi^\dagger(\vec{x}_n)|0\rangle\,=$$

$$=\frac{1}{\sqrt{n!}}\int[\mathrm{d}x]\sum_{i<j}^{n}V(\vec{x}_i,\vec{x}_j)\chi_n(\vec{x}_1,\dots\vec{x}_n,t)\,\Phi^\dagger(\vec{x}_1)\Phi^\dagger(\vec{x}_2)\dots\dots\Phi^\dagger(\vec{x}_n)|0\rangle\,.$$

Thus, we have shown that

$$\mathrm{H}|\chi\rangle_n=\frac{1}{\sqrt{n!}}\int[\mathrm{d}x]\,\Big(-\frac{\hbar^2}{2m}\sum_{i=1}^{n}\vec{\nabla}^2_{x_i}+\sum_{i<j}^{n}V(\vec{x}_i,\vec{x}_j)\Big)\chi_n(\vec{x}_1,\dots\vec{x}_n,t)\,\times$$
$$\times\,\Phi^\dagger(\vec{x}_1)\Phi^\dagger(\vec{x}_2)\dots\dots\Phi^\dagger(\vec{x}_n)|0\rangle\,.$$

Hence, if $\chi_n(\vec{x}_1,\dots\vec{x}_n,t)$ coincides with the wave function $\psi(\vec{x}_1,\dots,\vec{x}_n,t)$ evolving according to (3.44), we find that

$$\mathrm{H}|\chi\rangle_n=i\hbar\frac{\partial}{\partial t}|\chi\rangle_n\,.$$

In this way we have shown that the standard wave function description of a quantum mechanical model governed by the Schrödinger equation (3.44) is equivalent to evolution of the Fock space vector $|\chi\rangle_n$ under the second quantized Hamiltonian H.

Introduce the so-called *number operator*

$$N=\int \mathrm{d}x\,\Phi^\dagger(\vec{x})\Phi(\vec{x}) \tag{3.49}$$

which is indeed a dimensionless quantity. The characteristic feature of the Hamiltonian H is that it commutes with the number operator

$$[\mathrm{H}, N] = 0\,. \tag{3.50}$$

This means that the quantum-mechanical Hamiltonian preserves a number of particles and this is a reason why we are able to restrict our consideration to the n-particle sector of the Fock space.

Further, we can consider *the Heisenberg picture* where the operators are assumed to be time-dependent with the time evolution described by the Heisenberg equations of motion

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= \frac{i}{\hbar}[\mathrm{H}, \Phi]\,, \\ \frac{\partial \Phi^\dagger}{\partial t} &= \frac{i}{\hbar}[\mathrm{H}, \Phi^\dagger]\,. \end{aligned}$$

Explicitly, they are

$$\begin{aligned} i\hbar\frac{\partial \Phi(\vec{x})}{\partial t} &= -\frac{\hbar^2}{2m}\vec{\nabla}^2\Phi(\vec{x}) + \int \mathrm{d}\vec{y}\, V(\vec{x},\vec{y})\,\Phi^\dagger(\vec{y})\Phi(\vec{y})\,\Phi(\vec{x})\,, \\ i\hbar\frac{\partial \Phi^\dagger(\vec{x})}{\partial t} &= \frac{\hbar^2}{2m}\vec{\nabla}^2\Phi^\dagger(\vec{x}) - \Phi^\dagger(\vec{x})\int \mathrm{d}\vec{y}\, V(\vec{x},\vec{y})\,\Phi^\dagger(\vec{y})\Phi(\vec{y})\,. \end{aligned}$$

Now we would like to show that the second quantized Hamiltonian can be viewed as a coming from quantization of an underlying classical field theory. We can rescale $\Phi \to \sqrt{\hbar}\,\Phi$. The rescaled operators have the commutation relation

$$\big[\Phi(\vec{x}_1), \Phi^\dagger(\vec{x}_2)\big] = \hbar\,\delta(\vec{x}_1 - \vec{x}_2) \tag{3.51}$$

and their dimension is $[\Phi] = \hbar^{1/2}\ell^{-3/2}$. In terms of rescaled operators the Hamiltonian reads

$$\mathrm{H} = \frac{\hbar}{2m}\int \mathrm{d}x\,\partial_i\Phi^\dagger(\vec{x})\partial_i\Phi(\vec{x}) + \frac{1}{2\hbar^2}\int \mathrm{d}\vec{x}\mathrm{d}\vec{y}\;\Phi^\dagger(\vec{x})\Phi^\dagger(\vec{y})V(\vec{x},\vec{y})\Phi(\vec{x})\Phi(\vec{y})\,. \tag{3.52}$$

The advantage of this rescaling is that we can think of (3.51) as been result of quantization of the Poisson bracket for the complex scalar field $\Phi(x)$

$$\{\Phi(\vec{x}), \Phi^*(\vec{y})\} = i\delta(\vec{x} - \vec{y})\,. \tag{3.53}$$

The Hamiltonian and the Poisson bracket above define a *classical field theory* with the action

$$S[\Phi, \Phi^*] = \int \mathrm{d}\,\vec{x}\mathrm{d}t\,\Big(-i\Phi^*\partial_t\Phi\Big) - \int \mathrm{d}t\,\mathrm{H}\,, \tag{3.54}$$

where the Hamiltonian is

$$\mathrm{H} = \frac{\hbar}{2m}\int \mathrm{d}\vec{x}\,\partial_i\Phi^*(\vec{x})\partial_i\Phi(\vec{x}) + \frac{1}{2\hbar^2}\int \mathrm{d}\vec{x}\mathrm{d}\vec{y}\;\Phi^*(\vec{x})\Phi^*(\vec{y})V(\vec{x},\vec{y})\Phi(\vec{x})\Phi(\vec{y})\,. \tag{3.55}$$

Note that this action is of the *first* order in time derivative (this reflects its non-relativistic nature) and it has the proper physical dimension of the Planck constant $[S] = \hbar$. One can check that the classical equations of motion derived from the action $S[\Phi, \Phi^*]$ coincide with the Hamiltonian equations of motion

$$\frac{\partial \Phi}{\partial t} = \{\mathrm{H}, \Phi\}\,, \qquad \frac{\partial \Phi^*}{\partial t} = \{\mathrm{H}, \Phi^*\}\,.$$

One should not be confused by appearance of $\hbar$ in the classical action, the Planck constant is there for dimensional reasons, *i.e.* to provide the correct engineering dimension for the Hamiltonian and the action.

The Hamiltonian of the type (3.52) arises in the non-relativistic limit of the corresponding Hamiltonian for a complex Klein-Gordon field.

Show this!

3.5.2 Relevan formulae involving Bessel functions

To be extended

The Mehler-Sonine representation

$$K_0(z) = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}t\, e^{iz\sinh t} = \int_0^{\infty} \mathrm{d}t\, \cos(z\sinh t)\,. \tag{3.56}$$

$$H_\nu^{(1)}(z) = \frac{e^{-\frac{1}{2}i\pi\nu}}{i\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, e^{iz\cosh t-\nu t}\,, \tag{3.57}$$

$$H_\nu^{(2)}(z) = -\frac{e^{\frac{1}{2}i\pi\nu}}{i\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, e^{-iz\cosh t-\nu t}\,. \tag{3.58}$$

Analytic continuation

$$K_1(ix) = -\frac{\pi}{2}H_1^{(2)}(x)\,, \qquad x>0\,.$$

Chapter 4

Dirac field

I remember that when someone had started to teach me about creation and annihilation operators, that this operator creates an electron. I said 'How do you create an electron? It disagrees with the conservation of charge' .

Richard Feynman
Nobel Prize Lecture

Until now, everyone thought that the Dirac equation referred directly to physical particles. Now, in field theory, we recognize that the equations refer to a sublevel. Experimentally we are concerned with particles, yet the old equations describe fields.... When you begin with field equations, you operate on a level where the particles are not there from the start. It is when you solve the field equations that you see the emergence of particles.

Julian Schwinger

4.1 Introducing the Dirac equation

In 1928 Dirac discovered the relativistic equation which now bears his name trying to overcome the difficulties of negative probability densities of the Klein-Gordon equation. This equation has a special importance because it describes particles with spin $\frac{1}{2}$. The reasoning which led Dirac to his equation was as follows: If we wish to prevent the occurrence of negative probability densities, we must avoid time derivatives in the expression for ρ. The wave equation must therefore *not* contain time derivatives higher than the first order. Relativistic covariance, furthermore, requires that that there be essentially complete symmetry in the treatment of the spatial and time components. Thus the Dirac wave function must satisfy the first-order linear differential equation in all four coordinates. The linearity is required by the superposition principle of quantum mechanics. Finally, we also want that the wave function obeys the Klein-Gordon equation

$$\Big(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_i^2} + \frac{m^2c^2}{\hbar^2}\Big)\psi(x) = 0\,,$$

because in this case it describes a free relativistic particle with the dispersion relation

$$E^2 = p^2c^2 + m^2c^4\,.$$

Thus, the equation we are looking for could have the form

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\Big(\alpha_1\frac{\partial\psi}{\partial x_1} + \alpha_2\frac{\partial\psi}{\partial x_2} + \alpha_3\frac{\partial\psi}{\partial x_3}\Big) + \beta mc^2\psi \equiv \mathscr{H}\psi\,, \tag{4.1}$$

i.e. look similar to the standard Schrödinder equation containing the first time derivative of the wave function. What is quite striking is that the coefficients α_i entering this equation cannot be just numbers, as the equation would not be then even invariant under usual three-dimensional rotations. Also the wave function ψ cannot be just a scalar, because the probability density $\rho = \psi^*\psi$ must be a time component of the four vector.

This arguing led Dirac to consider equation (4.1) as a matrix equation. In analogy with a non-relativistic quantum mechanics of electron, the wave function is considered as a column with n components

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix},$$

while constant coefficients α_i and β are understood as $n \times n$ matrices. To proceed, we rewrite the equation (4.1) in the form

$$\Big(i\hbar\frac{\partial\psi}{\partial t} + i\hbar c\,\alpha_i\frac{\partial}{\partial x_i} - \beta mc^2\Big)\psi = 0$$

and act on this equation with the operator

$$i\hbar\frac{\partial\psi}{\partial t} - i\hbar c\,\alpha_i\frac{\partial}{\partial x_i} + \beta mc^2\,.$$

As a result, we get

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} + \hbar^2c^2\frac{\alpha_i\alpha_j + \alpha_j\alpha_i}{2}\frac{\partial^2\psi}{\partial x_i\partial x_j} - i\hbar mc^3(\alpha_i\beta + \beta\alpha_i)\frac{\partial^2\psi}{\partial x_i\partial t} - (mc^2)^2\beta^2\psi = 0\,.$$

We further multiply this equality with $-1/(c^2\hbar^2)$ getting thereby

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{\alpha_i\alpha_j + \alpha_j\alpha_i}{2}\frac{\partial^2\psi}{\partial x_i\partial x_j} + i\frac{mc}{\hbar}(\alpha_i\beta + \beta\alpha_i)\frac{\partial^2\psi}{\partial x_i\partial t} + \frac{m^2c^2}{\hbar^2}\beta^2\psi = 0\,.$$

Thus, we see that ψ would satisfy the Klein-Gordon equation, provided the following relations are satisfied

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}\,, \qquad \alpha_i\beta + \beta\alpha_i = 0\,, \qquad \beta^2 = \mathbb{1}\,.$$

In addition to these conditions on the matrices α_i and β we require that they are hermitian, in which case the Dirac Hamiltonian $\mathscr{H}$ in equation (4.1) will be a hermitian operator. Further, from the commutation relations we see that $\alpha_i^2 = \beta^2 = \mathbb{1}$, so eigenvalues of all the matrices are just $+1$ or -1. From $\alpha_i = -\beta\alpha_i\beta$ we conclude that $\mathrm{Tr}\,\alpha_i = 0$. Analogously,

$$\mathrm{Tr}\,\beta = \mathrm{Tr}\,(\alpha_i^2\beta) = \mathrm{Tr}\,(\alpha_i\beta\alpha_i) = -\mathrm{Tr}\,\beta\,,$$

so that $\mathrm{Tr}\,\beta = 0$. This implies that the number of eigenvalues equal to $+1$ must coincide with the number of eigenvalues equal to -1, that is the number n defining the size of the matrices α_i, β must be even. In the minimal dimension $n = 2$, we find the three Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.2}$$

which are hermitian, traceless and obey the desired relation $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$. However, for $n = 2$ the fourth independent matrix is just an identity matrix[1], which cannot be identified with β. It appears that the minimal dimension where one can construct the four matrices α_i, β with the desired properties is equal to four. In one concrete representation, which matrices look as follows

$$\alpha_i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{4.3}$$

4.2 The Dirac equation and Lorentz transformations

To discuss covariant properties of the Dirac equation it is desirable to rewrite the Dirac equation in the covariant four-component form which takes into account the symmetry between time and space. To do this, we multiply (4.2) by β/c and introduce the notation

$$\gamma^0 = \beta\,, \qquad \gamma^i = \beta\alpha_i\,, \qquad i = 1, 2, 3\,. \tag{4.4}$$

and taking into account that $x^0 = ct$ and $\frac{\partial}{\partial x^0} = \frac{\partial}{c\partial t}$, we obtain

$$\Big[i\hbar\Big(\gamma^0\frac{\partial}{\partial x^0} + \gamma^i\frac{\partial}{\partial x^i}\Big) - mc\Big]\psi = 0\,,$$

or, more elegantly,

$$\Big(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\Big)\psi = 0\,. \tag{4.5}$$

In natural units $\hbar = c = 1$ this equation reads

$$\Big(i\gamma^\mu\partial_\mu - m\Big)\psi = 0\,. \tag{4.6}$$

The newly introduced matrices γ^μ satisfy the following commutation relations[2]

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,\mathbb{1}\,. \tag{4.7}$$

The free algebra generated by the symbols γ^μ modulo the relation (4.7) is called the *Clifford algebra*. From the definition (4.4) it is clear that γ^0 is hermitian and all γ^i are anti-hermitian. In our representation the γ-matrices look like

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{4.8}$$

This representation of γ-matrices in which γ^0 is diagonal is called *the Dirac representation*.

Now we investigate properties of the γ-matrices a bit further. From four γ^μ one can construct 16 linearly independent matrices

[1] Together with the identity matrix the Pauli matrices form a complete set of linearly independent matrices over which any hermitian matrix can be decomposed.

[2] Indeed, one can easily verify, for instance, that $\gamma^i\gamma^j + \gamma^j\gamma^i = \beta\alpha_i\beta\alpha_j + \beta\alpha_j\beta\alpha_i = -\alpha_i\alpha_j - \alpha_j\alpha_i = -2\delta_{ij}$, and $\gamma^i\gamma^0 + \gamma^0\gamma^i = \beta\alpha_i\beta + \beta\beta\alpha_i = -\alpha_i + \alpha_i = 0$.

- identity matrix $\mathbb{1} = g^{\mu\mu}\gamma^\mu\gamma^\mu$ (no summation in μ);
- four matrices $\gamma^\mu, \quad \mu = 0, \ldots, 3$;
- six matrices $\sigma^{\mu\nu} = i\frac{\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu}{2} = i\gamma^\mu\gamma^\nu, \quad \mu < \nu; \ \mu, \nu = 0, \ldots, 3$;
- one matrix $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$;
- four matrices $\tau^\mu = \gamma^\mu\gamma^5, \quad \mu = 0, \ldots, 3.$

It is not difficult to see that all other possible products of γ-matrices and their liner combinations are expressible via the sixteen matrices introduced above by means of lineal relations. To show that these sixteen matrix are linear independent, we first prove that they all have vanishing traces. Consider for instance γ^μ. From (4.7) we see that $\gamma_\nu\gamma^\nu = 1$, where ν is fixed and there is no summation in ν. Then

$$\mathrm{Tr}\gamma^\mu = \mathrm{Tr}(\gamma_\nu\gamma^\nu\gamma^\mu)\,|_{\nu\neq\mu} = \tfrac{1}{2}\mathrm{Tr}\gamma_\nu(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\,|_{\nu\neq\mu} = 0\,.$$

Analogously, one can show that the traces of all other matrices are zero. Exercise for the reader – show this! In particular $\mathrm{Tr}\gamma^5 = 0$. The fact that traces of all sixteen matrices, except the identity matrix, is zero can be used to show that they are linearly independent. Assume the opposite. Let

$$F = a\mathbb{1} + b^\mu\gamma^\mu + c^{\mu\nu}\sigma^{\mu\nu} + d^\mu\tau^\mu + e\gamma^5\,,$$

where a, b, c, d, e are arbitrary complex coefficients. If all matrices are linearly independent then F must be equal to zero. Taking trace of F we find $a = 0$. Let us now consider $\gamma^\nu F$ and take the trace of this expression, this will give $b^\nu = 0$. Proceeding in this way, one can show that all coefficients in F are zero, *i.e.* all the sixteen matrices are linearly independent.

Note that the Clifford algebra relations, as well as all the relations derived from them above, are invariant with respect to the unitary transformations

$$\gamma^\mu \to O\gamma^\mu O^{-1}\,.$$

Now we are ready to show the covariance of the Dirac equation with respect to the Lorentz transformations. Assume that under the Lorentz transformation

$$x'^\mu = \Lambda^\mu_\nu x^\nu\,, \qquad \eta_{\mu\nu}\Lambda^\mu_\alpha\Lambda^\nu_\beta = \eta_{\alpha\beta}$$

the object $\psi(x)$ transforms as follows $\psi(x) \to \psi'(x') = S(\Lambda)\psi(x)$, where $S(\Lambda)$ is a matrix which depends of course on the transformation matrix Λ. We recall that the general form of Λ is

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & -\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{v^t}{c} \\ -\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{v}{c} & 1 + \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)\frac{v\otimes v^t}{v^2} \end{pmatrix},$$

where R is a three-dimensional rotation and v is a velocity of the boost.

The relativity principle asserts that in the new Lorentz frame the Dirac equation must look the same, that is

$$\left(i\gamma^\mu\frac{\partial}{\partial x'^\mu} - \frac{mc}{\hbar}\right)\psi'(x') = 0\,.$$

Note that

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = \Lambda^\nu_\mu\frac{\partial}{\partial x'^\nu} \quad \text{or} \quad \Lambda^\nu_\mu\frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial x'^\mu}\,.$$

Substituting $\psi'(x')$ and multiplying the Dirac equation from the left with $S^{-1}(\Lambda)$, we get

$$\Big(iS^{-1}(\Lambda)\gamma^\mu S(\Lambda)\frac{\partial}{\partial x'^\mu}-\frac{mc}{\hbar}\Big)\psi(x)=\Big(iS^{-1}(\Lambda)\gamma^\mu S(\Lambda)\Lambda^\nu_\mu\frac{\partial}{\partial x^\nu}-\frac{mc}{\hbar}\Big)\psi(x)=0\,.$$

Thus, the covariance of the equation requires

$$S^{-1}\gamma^\mu S\,\Lambda^\mu_\nu=\gamma^\mu\,,$$

which is the same as

$$S^{-1}\gamma^\mu S\,=\Lambda^\mu_\nu\gamma^\nu\,. \tag{4.9}$$

To solve this equation for S, we first attempt an infinitezimal analysis, that is we assume that S is of the form

$$S=\mathbb{1}+\lambda^{\mu\nu}\omega_{\mu\nu}+\dots\,,$$

where $\Lambda_{\mu\nu}=g_{\mu\nu}+\omega_{\mu\nu}+\dots$ and $\omega_{\mu\nu}=-\omega_{\nu\mu}$. Thus, at leading order we find the following eqiuation

$$\gamma^\mu\lambda^{\rho\lambda}-\lambda^{\rho\lambda}\gamma^\mu=g^{\mu[\rho}\gamma^{\lambda]}=g^{\mu[\rho}\gamma^{\lambda]}=\tfrac{1}{2}(g^{\mu\rho}\gamma^\lambda-g^{\mu\lambda}\gamma^\rho)\,. \tag{4.10}$$

This is solved by

$$\lambda^{\mu\nu}=\frac{1}{8}\gamma^{[\mu}\gamma^{\nu]}=\frac{1}{4}\gamma^\mu\gamma^\nu\,. \tag{4.11}$$

The finite transformation would have the form

$$S=\exp\Big(\frac{1}{4}\sum_{\mu,\nu=0}^{3}\gamma^\mu\gamma^\nu\omega_{\mu\nu}\Big)=\exp\Big(\frac{1}{2}\sum_{\mu<\nu}\gamma^\mu\gamma^\nu\omega_{\mu\nu}\Big)\,. \tag{4.12}$$

Let ϕ denotes a usual rotation in the plane ij, then one can show that the corresponding 4×4 matrix S has the form

$$S^{ij}(\phi)=\exp(\tfrac{1}{2}\gamma^i\gamma^j\phi)=\mathbb{1}\,\cos\frac{\phi}{2}+\gamma^i\gamma^j\sin\frac{\phi}{2}\,. \tag{4.13}$$

For the Lorentz boosts we obtain

$$S^{i}(\phi)=\exp(\tfrac{1}{2}\gamma^0\gamma^i\phi)=\mathbb{1}\,\cosh\frac{\phi}{2}+\gamma^0\gamma^i\sinh\frac{\phi}{2}\,. \tag{4.14}$$

It is clear from formula (4.13) after the rotation by the angle 2π the coordinate system turns into itself, while $S^{ij}(2\pi)=-1$, that is the spinor ψ will change its overall sign. Thus, the spinor itself cannot be an observable quantity, but its quadratic combinations are.

Notice that the rotations in ij-plane we can write in the 2×2 block-diagonal form[3]

$$S^{ij}(\phi)=\begin{pmatrix}\mathbb{1}\cos\frac{\phi}{2}-i\epsilon^{ijk}\sigma^k\sin\frac{\phi}{2} & 0\\ 0 & \mathbb{1}\cos\frac{\phi}{2}-i\epsilon^{ijk}\sigma^k\sin\frac{\phi}{2}\end{pmatrix}\,. \tag{4.15}$$

Hence, we we split the spinor into two two-component spinors as

$$\psi=\begin{pmatrix}\phi\\ \chi\end{pmatrix}\,, \tag{4.16}$$

[3]We use that $\sigma^i\sigma^j=\mathbb{1}\delta^{ij}+i\epsilon^{ijk}\sigma^k$ and the fact that S^{ij} is defined for $i\neq j$.

then under rotations ϕ and ψ transform completely independently, each of them realize in fact an irreducible spinor representation of the rotation group SO(3).

For space rotations

$$S^\dagger = \exp(\tfrac{1}{2}\sum_{i<j}\gamma^{j\dagger}\gamma^{i\dagger}\omega_{ij}) = \exp(\tfrac{1}{2}\sum_{i<j}\gamma^{j}\gamma^{i}\omega_{ij}) = \exp(-\tfrac{1}{2}\sum_{i<j}\gamma^{i}\gamma^{j}\omega_{ij}) = S^{-1}\,,$$

because γ^i are anti-hermitian. Thus, for space rotations S is unitary: $S^\dagger S = \mathbb{1}$. However, for boosts we find

$$S^\dagger = \exp(\tfrac{1}{2}\gamma^{i\dagger}\gamma^{0\dagger}\omega_{0i}) = \exp(-\tfrac{1}{2}\gamma^{i}\gamma^{0}\omega_{0i}) = \exp(\tfrac{1}{2}\gamma^{0}\gamma^{i}\omega_{0i}) = S\,,$$

as γ^0 is hermitian. Both formulae can be combined in one

$$S^\dagger = \gamma^0 S^{-1}\gamma^0\,.$$

This motivates to introduce a *Dirac* conjugate spinor

$$\bar\psi = \psi^{*t}\gamma^0\,. \tag{4.17}$$

Under Lorentz transformations it will transform as follows

$$\overline{\psi'(x')} = (S\psi)^{*t}\gamma^0 = \psi^{*t}S^\dagger\gamma^0 = \psi^{*t}\gamma^0\gamma^0 S^\dagger\gamma^0 = \bar\psi S^{-1}\,. \tag{4.18}$$

This allows us to deduce that the bilinear combination like $\bar\psi\psi$ is a Lorentz scalar.

We have not yet found the transformation of the spinor field with respect to the parity transformation $\mathscr{P}$: $\vec{x}\to-\vec{x}, t\to t$. The Dirac equation will remain invariant under this transformation if we requite that the parity operation $\mathscr{P}$ is realized as a multiplication of a spinor by a matrix P

$$\psi'(x') = P\psi(x)\,,$$

where P should obey the relations

$$P^{-1}\gamma^0 P = \gamma^0\,,\qquad P^{-1}\gamma^i P = -\gamma^i\,, \tag{4.19}$$

which are satisfied by taking $P=\eta_p\gamma^0$. Since P is required to be a unitary operator $P^\dagger P=\mathbb{1}$, η_p must be a phase $|\eta_p|=1$. The eigenvalues of P are determined in the usual way, that is from the condition that a parity transformation applied twice is equivalent to the identity: $P^2=\mathbb{1}$. However, for a spinor the double parity transformation can be considered as a rotation on the angle 2π, where the spinor change the sign. Thus, two alternative definitions of parity operation are possible

$$P^2=\mathbb{1}\quad\text{or}\quad P^2=-\mathbb{1}\,.$$

Thus, the eigenvalues can be

$$\eta_p=\pm1\quad\text{or}\quad\eta_p=\pm i\,.$$

The number η_p is called an internal parity. Note also that P anti-commutes with the matrix γ^5: $[P,\gamma^5]=0$. If we act on ψ written in the form (4.16) with P we get

$$P\psi=\eta\gamma^0\psi=\begin{pmatrix}\eta_p\phi\\-\eta_p\chi\end{pmatrix}\,,$$

which shows that ϕ and χ form in fact two spinor representations of the rotation group O(3) but of opposite internal parity.

Analogously, one can show that the following bilinear combinations transform under Lorentz transformations as

- $\bar{\psi}\psi$ – a scalar;
- $\bar{\psi}\gamma^5\psi$ – a pseudo-scalar;
- $\bar{\psi}\gamma^\mu\psi$ – a vector;
- $\bar{\psi}\gamma^5\gamma^\mu\psi$ – a pseudo-vector vector.

4.3 On various representations of the Dirac equation

If we transform the spinor as

$$\psi \to \psi' = O\psi$$

and simultaneously transform $\gamma^\mu \to \gamma'^\mu = O\gamma^\mu O^{-1}$ then the Dirac equation remains invariant and the Clifford algebra does not change its form

$$\gamma'^\mu\gamma'^\nu + \gamma'^\nu\gamma'^\mu = 2g^{\mu\nu}\mathbb{1}\,. \tag{4.20}$$

Thus, using this freedom one can choose various representations for gamma-matrices. One of them, *the Dirac representation*, we have already introduced

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{4.21}$$

Another convenient representation is the so-called *Weyl or chiral representation*

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{4.22}$$

In this representation all gamma-matrices are off-diagonal. The transformation is given by

$$O\gamma_d O^{-1} = \gamma_c\,, \qquad O = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}, \quad O^\dagger O = \mathbb{1}\,,$$

where γ_d and γ_c are the gamma-matrices in the Dirac and chiral representation, respectively. In the chiral basis the equations for the two-component spinors take the form

$$\frac{1}{c}\frac{\partial\phi}{\partial t} - \sigma^i\frac{\partial\phi}{\partial x_i} + i\frac{mc}{\hbar}\chi = 0\,. \tag{4.23}$$

$$\frac{1}{c}\frac{\partial\chi}{\partial t} + \sigma^i\frac{\partial\chi}{\partial x_i} + i\frac{mc}{\hbar}\phi = 0 \tag{4.24}$$

Since the matrices γ^i for $i = 1, 2, 3$ are the same in both the Dirac and the chiral representations, ϕ and χ continue to transform independently and irreducibly under the action of the rotation group SO(3). However, a new remarkable fact is now that ϕ and χ transform independently under proper Lorentz transformations! Indeed, the matrix $S^i(\phi)$ takes in the chiral basis the following form

$$S^i(\phi) = \begin{pmatrix} \mathbb{1}\cosh\frac{\phi}{2} - \sigma^i\sinh\frac{\phi}{2} & 0 \\ 0 & \mathbb{1}\cosh\frac{\phi}{2} + \sigma^i\sinh\frac{\phi}{2} \end{pmatrix}. \tag{4.25}$$

In reality one should not be surprised that the representation of the *proper* Lorentz group on four-dimensional spinors appear to be reducible. Since γ^5 anti-commutes with all gamma-matrices, it commutes with a matrix of Lorentz transformations

$$[\gamma^5, S(\Lambda)] = 0\,,$$

which precisely means that the representation is reducible. It is just that in the Weyl basis this fact becomes obvious. We see however that on two-dimensional spinors we have *two different representations* of the Lorentz group. These two-dimensional representations are inequivalent. Indeed, equivalence would mean an existence of an invertible operator V, such that

$$V\Big(\mathbb{1}\cosh\frac{\phi}{2}-\sigma^i\sinh\frac{\phi}{2}\Big)V^{-1}=\mathbb{1}\cosh\frac{\phi}{2}+\sigma^i\sinh\frac{\phi}{2}\,,$$

which implies that

$$V\sigma^iV^{-1}=-\sigma^i\,,\quad \text{for all}\quad i=1,2,3.$$

It is easy to see that a matrix which must anti-commute with all three Pauli matrices does not exist.

If mass vanishes then eqs.(4.23) split into two independent (uncoupled) equations for ϕ and ψ, which are known as the Weyl equations. Thus, when particle of spin 1/2 is massless it can be consistently described by one two-component spinor, satisfying the Weyl equation. The Weyl equation is covariant with respect to the proper Lorentz group but it is not invariant under parity.

4.4 Solution of the Dirac equation

We start with the Dirac equation

$$\Big(i\gamma^\mu\partial_\mu-\frac{mc}{\hbar}\Big)\psi=0\,. \tag{4.26}$$

The general solution can be represented as the Fourier integral

$$\psi(\vec{x},t)=\int\frac{\mathrm{d}\vec{p}}{(2\pi\hbar)^3}\,e^{-\frac{i}{\hbar}Et+\frac{i}{\hbar}\vec{p}\vec{x}}\psi(p) \tag{4.27}$$

Substitution gives

$$\Big(E\gamma^0-c\,\gamma^ip_i-mc^2\mathbb{1}\Big)\psi(p)=0\,. \tag{4.28}$$

Multiplying this equation from the left by γ^0, we see that it is equivalent to the eigenvalue problem for the Dirac Hamiltonian $\mathscr{H}$

$$\mathscr{H}\psi=\Big(c\,\alpha_ip_i+mc^2\beta\Big)\psi=E\psi\,.$$

In terms of two-component spinors equation (4.28) splits into two

$$\begin{aligned}(E-mc^2)\phi-c\,\vec{\sigma}\vec{p}\,\chi&=0\\ -c\,\vec{\sigma}\vec{p}\,\phi+(E+mc^2)\chi&=0\,.\end{aligned}$$

This system can be solved provided the following determinant vanishes

$$\begin{vmatrix}E-mc^2 & -c\,\vec{\sigma}\vec{p}\\ -c\,\vec{\sigma}\vec{p} & E+mc^2\end{vmatrix}=E^2-(mc^2)^2-c^2\vec{p}^2=0\,.$$

Thus, we again rediscover the dispersion relation of the relativistic particle which has two solutions

$$E=\pm\sqrt{\vec{p}^2c^2+m^2c^4}\,.$$

That is for a fixed momentum $\vec{p}$ the Dirac Hamiltonian has solutions with positive and negative energy. If the sign of the energy is chosen, then one of the two-component spinors can be expressed via the other, for instance,

$$\psi=\begin{pmatrix}\phi\\ \frac{c\,\vec{\sigma}\vec{p}}{E+mc^2}\phi\end{pmatrix}. \tag{4.29}$$

Figure 4.1: Helicity. In the first picture the spin and velocity of a particle are parallel (helicity +1). In the second picture the spin and velocity are anti-parallel (helicity -1). Neutrino is left-handed while anti-neutrino is right-handed.

We therefore consider now the positive and negative energy solutions

$$u_+ = \begin{pmatrix} \phi \\ \frac{c\,\vec{\sigma}\vec{p}}{E(p)+mc^2}\phi \end{pmatrix}, \qquad u_- = \begin{pmatrix} \phi \\ \frac{c\,\vec{\sigma}\vec{p}}{-E(p)+mc^2}\phi \end{pmatrix}, \tag{4.30}$$

where in both formulas $E(p) = \sqrt{\vec{p}^2c^2 + m^2c^4}$ is a positive expression. Since ϕ remains an arbitrary two-component spinor, we have two independent solutions with positive energy and two independent solutions with negative energy.

Existence of two independent solutions for each sign of energy can be explained by the fact that there exist an operator which commutes with the Dirac Hamiltonian. It is a helicity operator given by

$$\mathscr{S} = \frac{1}{|\vec{p}|}\gamma^0\gamma^5\gamma^i p_i = \frac{1}{|\vec{p}|}\begin{pmatrix} \vec{\sigma}\vec{p} & 0 \\ 0 & \vec{\sigma}\vec{p} \end{pmatrix}.$$

Since $[\mathscr{H}, \mathscr{S}] = 0$, these two operators can be simultaneously diagonalized. Further, due to the fact that $\mathscr{S}^2 = \mathbb{1}$, eigenvalues of $\mathscr{S}$ are ± 1. Thus, a state with positive energy can be also an eigenstate of the helicity operator with either positive or negative helicity, and the same holds for states with negative energy. We first present the corresponding positive energy solutions (with for the moment undetermined overall normalization $\mathcal{N}$)

$$u_+^1 = \mathcal{N}\begin{pmatrix} p_3 + |\vec{p}| \\ p_1 + ip_2 \\ \frac{c|\vec{p}|(p_3+|\vec{p}|)}{mc^2+E(p)} \\ \frac{c|\vec{p}|(p_1+ip_2)}{mc^2+E(p)} \end{pmatrix}, \qquad u_+^2 = \mathcal{N}\begin{pmatrix} -(p_1 - ip_2) \\ p_3 + |\vec{p}| \\ \frac{c|\vec{p}|(p_1-ip_2)}{mc^2+E(p)} \\ -\frac{c|\vec{p}|(p_3+|\vec{p}|)}{mc^2+E(p)} \end{pmatrix}. \tag{4.31}$$

Here u_+^1 has the helicity $+1$ and u_+^2 has the helicity -1, respectively. Analogously, solutions with negative energy are

$$u_-^1 = \mathcal{N}\begin{pmatrix} -\frac{c|\vec{p}|(p_3+|\vec{p}|)}{mc^2+E(p)} \\ -\frac{c|\vec{p}|(p_1+ip_2)}{mc^2+E(p)} \\ p_3 + |\vec{p}| \\ p_1 + ip_2 \end{pmatrix}, \qquad u_-^2 = \mathcal{N}\begin{pmatrix} -\frac{c|\vec{p}|(p_1-ip_2)}{mc^2+E(p)} \\ \frac{c|\vec{p}|(p_3+|\vec{p}|)}{mc^2+E(p)} \\ -(p_1 - ip_2) \\ p_3 + |\vec{p}| \end{pmatrix}. \tag{4.32}$$

Here u^1_- has the helicity $+1$ and u^2_- has the helicity -1, respectively. Having these explicit solutions, one can see that it is possible to chose a normalization, namely,

$$\mathcal{N} = \frac{1}{2}\sqrt{\frac{mc^2 + E(p)}{mc^2(\vec{p}^2 + p_3|\vec{p}|)}}$$

such that

$$\bar{u}^r_+(p)u^s_+(p) = (u^r_+(p))^*\gamma^0 u^s_+(p) = \delta^{rs}\,, \qquad \bar{u}^r_-(p)u^s_-(p) = (u^r_-(p))^*\gamma^0 u^s_-(p) = -\delta^{rs}\,, \qquad r,s = 1,2\,.$$

The fact that spinors of different helicity are orthogonal follows from the fact they belong to different eigenvalues of the helicity operator $\mathscr{S}$ and also from the fact that $[\mathscr{S},\gamma^0] = 0$. However, since $[\mathscr{H},\gamma^0] \neq 0$, in general $\bar{u}^r_+ u^s_- \neq 0$. However, if we also change the sign of momentum in one of the spinors, then the orthogonality holds

$$\bar{u}^r_+(p)u^s_-(-p) = 0\,, \qquad r,s,= 1,2.$$

The proof of this relation is based on the identity

$$\gamma^0\mathscr{H}(p)\gamma^0 = \mathscr{H}(-p)\,.$$

We have

$$\mathscr{H}(p)u_+(p) = E(p)u_+(p) \quad \Longrightarrow \quad u^\dagger_+(p)\mathscr{H}(p) = E(p)u^\dagger_+(p)\,,$$

giving

$$\bar{u}_+(p)\underbrace{\gamma^0\mathscr{H}(p)\gamma^0}_{\mathscr{H}(-p)} = E(p)\bar{u}_+(p)\,.$$

Multiplying the last formula by $u_(-p)$ from the right, we get

$$\bar{u}_+(p)\underbrace{\mathscr{H}(-p)u_-(-p)}_{-E(p)u_-(-p)} = E(p)\,\bar{u}_+(p)u_-(-p)\,.$$

Thus, we get a contradiction unless $\bar{u}_+(p)u_-(-p) = 0$. Similarly, one shows that $\bar{u}_-(p)u_+(-p) = 0$. We denote two linearly independent orthogonal negative energy solutions for momentum $-\vec{p}$ by $v^s_-(p)$. For reader's convenience, we give here a complete set of orthonormality relations

$$\begin{aligned} &\bar{u}^r_+(p)u^s_+(p) = \delta^{rs}\,, \qquad &&\bar{v}^r_-(p)v^s_-(p) = -\delta^{rs}\,,\\ &\bar{v}^r_-(p)u^s_+(p) = 0\,, \qquad &&\bar{u}^r_+(p)v^s_-(p) = 0\,. \end{aligned} \tag{4.33}$$

There is also another important issue related to the solutions of the Dirac equations we found. Introduce the following operators[4]

$$\begin{aligned} \Lambda_+(\vec{p}) &= \frac{c\not{p} + mc^2\mathbb{1}}{2mc^2} \equiv \frac{\gamma^0 E(p) - c\gamma^i p^i + mc^2\mathbb{1}}{2mc^2}\,,\\ \Lambda_-(\vec{p}) &= \frac{mc^2\mathbb{1} - c\not{p}}{2mc^2} \equiv \frac{mc^2\mathbb{1} - \gamma^0 E(p) + c\gamma^i p^i}{2mc^2}\,. \end{aligned} \tag{4.34}$$

It is not difficult to see that $\Lambda_\pm(\vec{p})$ are projectors and they provide an orthogonal decomposition of the unity

$$\Lambda^2_+ = \Lambda_+\,, \quad \Lambda^2_- = \Lambda_-\,, \quad \Lambda_+\Lambda_- = 0\,, \quad \Lambda_+ + \Lambda_- = \mathbb{1}\,.$$

[4]Here we introduced the standard notation $\not{p}$ which means $\not{p} = \gamma^\mu p_\mu = \gamma^0\vec{p}^0 - \gamma^i\vec{p}^i = \frac{1}{c}\gamma^0 E - \gamma^i\vec{p}^i$.

Projectors Λ_+ and Λ_- project on the solutions with positive and negative energy, that is

$$\begin{aligned}\Lambda_+(\vec{p})\, u_+^{1,2}(p) &= u_+^{1,2}(p)\,,\\ \Lambda_-(\vec{p})\, v_-^{1,2}(p) &= v_-^{1,2}(p)\,.\end{aligned} \tag{4.35}$$

It can be further found that the following relations are true

$$\begin{aligned}\Lambda_+(\vec{p})_{ij} &= \sum_{r=1}^{2}(u_+^r)_i(\bar{u}_+^r)_j = \sum_{r=1}^{2}(u_+^r)_i((u_+^r)^\dagger\gamma^0)_j\,,\\ \Lambda_-(\vec{p})_{ij} &= -\sum_{r=1}^{2}(v_-^r)_i(\bar{v}_-^r)_j = -\sum_{r=1}^{2}(v_-^r)_i((v_-^r)^\dagger\gamma^0)_j\,.\end{aligned} \tag{4.36}$$

One remark is in order. For massive particle the helicity operator does not commute with Lorentz transformations, *i.e.* the notion of helicity (that is being right-handed or left-handed) does depend on a choice of the Lorentz frame. By changing the Lorentz frame right-handed particle might become a left-handed one. At first it might seem surprizing because we solved the Dirac equation and associated to each solution a definite helicity. However, one has to realize that the Dirac Hamiltonian itself does not commute with Lorentz transformations – it represents energy which changes under the Lorentz transformations. Thus, fixing $\mathscr{H}$ we are not allowed anymore to perform Lorentz transformations, and, therefore, the helicity can be associated to the eigenstates of the Hamiltonian.[5]

Thus, an expansion of a Dirac field $\psi(x)$ over a complete set of orthogonal solutions is given by

$$\psi(x) = \int \frac{\mathrm{d}\vec{p}}{(2\pi\hbar)^{3/2}}\left(\frac{mc}{E(p)}\right)^{1/2}\sum_{r=1}^{2}\left(b_r(p)u_+^r(p)e^{-\frac{i}{\hbar}p_\mu x^\mu} + d_r^*(p)v_-^r(p)e^{\frac{i}{\hbar}p_\mu x^\mu}\right). \tag{4.37}$$

To check the correctness of this expression, we act on $\psi(x)$ with the Dirac operator $i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}$ and taking into account that

$$\begin{aligned}\Big(i\gamma^0\frac{\partial}{c\partial t} + i\gamma^i\frac{\partial}{\partial x^i} - \frac{mc}{\hbar}\Big)e^{-\frac{i}{\hbar}(Et-\vec{p}\vec{x})} &= \frac{\gamma^0}{c\hbar}\Big(E(p) - \mathscr{H}(p)\Big)e^{-\frac{i}{\hbar}(Et-\vec{p}\vec{x})}\,,\\ \Big(i\gamma^0\frac{\partial}{c\partial t} + i\gamma^i\frac{\partial}{\partial x^i} - \frac{mc}{\hbar}\Big)e^{+\frac{i}{\hbar}(Et-\vec{p}\vec{x})} &= -\frac{\gamma^0}{c\hbar}\Big(E(p) + \mathscr{H}(-p)\Big)e^{+\frac{i}{\hbar}(Et-\vec{p}\vec{x})}\end{aligned}$$

obtain $\Big(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\Big)\psi = 0$. The normalization prefactor in the Fourier transform is chosen for the later convenience.

We finish the section with the following comment. The helicity operator is not the only one which commutes with the Dirac Hamiltonian. Another important operator is the total momentum (that is the orbital angular momentum plus spin)

$$\mathscr{J}_i = i\epsilon_{ijk}x_jp_k + \frac{\hbar}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}.$$

Here x_i and p_i are the usual operators of coordinate and momentum. One verifies that $[\mathscr{H}, \mathscr{J}_i] = 0$.

4.5 Charge conjugation and anti-particles

So far our discussion of the Dirac equation, which was originally designed to described electrons, did not involve the electric charge. The point is that the properties of electron concerning its electric

[5]It is a general statement that passing to the Hamiltonian formalism one breaks the Lorentz invariance, since one fixes the preferred time direction.

charge show up when we put an electron in an external electromagnetic field. The Dirac equation which describes electrons coupled to an external electromagnetic field has the form

$$\Big[\gamma^\mu\Big(i\hbar\,\partial_\mu - \frac{e}{c}A_\mu\Big) - mc\Big]\psi = 0\,. \tag{4.38}$$

This equation has a new fundamental symmetry which is the symmetry of the theory with respect to the change of the sign of the electric charge, also considered as replacement of a particle by its anti-particle. This symmetry is known under the name *the charge conjugation*. This symmetry states that there is a one-to-one correspondence between solutions of the Dirac equation (4.38) with a *negative energy* and the wave function of a positron (that is a particle which have the same mass as electron but opposite sign electric charge). According to physical requirements, we regard a wave function of a positron ψ^c as a *positive energy solution* of an equation

$$\Big[\gamma^\mu\Big(i\hbar\,\partial_\mu + \frac{e}{c}A_\mu\Big) - mc\Big]\psi^c = 0\,. \tag{4.39}$$

Since sing of e played no role so far we can ask a general question about the existence of a transformation which transforms (4.38) into (4.39) and vice versa. We see that to change the relative sign between $i\hbar\,\partial_\mu$ and $\frac{e}{c}A_\mu$ it is enough to apply the complex conjugation to (4.38). This gives

$$\Big[(\gamma^\mu)^*\Big(i\hbar\,\partial_\mu + \frac{e}{c}A_\mu\Big) + mc\Big]\psi^* = 0\,. \tag{4.40}$$

Assuming an existence of a matrix C which provides the relation $\psi^* = (C\gamma^0)^{-1}\psi^c$, we bring the equation above to the form

$$\Big[(C\gamma^0)\gamma^{\mu *}(C\gamma^0)^{-1}\Big(i\hbar\,\partial_\mu + \frac{e}{c}A_\mu\Big) + mc\Big]\psi^c = 0\,. \tag{4.41}$$

Thus, if we subject C to the condition

$$(C\gamma^0)\gamma^{\mu *}(C\gamma^0)^{-1} = -\gamma^\mu \quad \text{or} \quad (C\gamma^0)^{-1}\gamma^\mu(C\gamma^0) = -\gamma^{\mu *}\,,$$

we would obtain the desired transformation. Under the map $\gamma^\mu \to -\gamma^{\mu *}$ the Clifford algebra relations are invariant and, therefore, such a matrix C must exist. We simply give an answer

$$C = i\gamma^2\gamma^0\,,$$

where C has the following properties

$$C^2 = -\mathbb{1}\,, \quad C^* = C\,, \quad C^\dagger C = \mathbb{1}\,.$$

Thus,

$$\psi^c = C\gamma^0\psi^* = i\gamma^2\psi^* = C\overline{\psi}^t\,, \tag{4.42}$$

as $(\gamma^0)^t = \gamma^0$. By using explicit expressions for $u^r_\pm$ is is easy to establish that

$$\begin{aligned}
&\mathscr{H}(p)(Cu^{1*}_+) = -E(p)(Cu^{1*}_+)\,, \qquad \mathscr{H}(p)(Cu^{1*}_-) = E(p)(Cu^{1*}_-)\,,\\
&\mathscr{H}(p)(Cu^{2*}_+) = -E(p)(Cu^{2*}_+)\,, \qquad \mathscr{H}(p)(Cu^{2*}_-) = E(p)(Cu^{2*}_-)\,,
\end{aligned} \tag{4.43}$$

as well as

$$\begin{aligned}
&\mathscr{S}(Cu^{1*}_+) = -(Cu^{1*}_+)\,, \qquad \mathscr{S}(Cu^{1*}_-) = -(Cu^{1*}_-)\,,\\
&\mathscr{S}(Cu^{2*}_+) = (Cu^{2*}_+)\,, \qquad \mathscr{S}(Cu^{2*}_-) = (Cu^{2*}_-)\,.
\end{aligned} \tag{4.44}$$

This gives a hint that the following relations hold, as one can verify by explicit calculation,

$$\begin{array}{ll} Cu_+^{1*}(p) = u_-^2(p)\,, & Cu_+^{2*}(p) = -u_-^1(p)\,, \\ Cv_-^{1*}(p) = u_+^2(-p)\,, & Cv_-^{2*}(p) = -u_+^1(-p)\,. \end{array} \tag{4.45}$$

Thus, formally the Dirac theory is invariant under the following sequence of operations

1) Complex conjugate;

2) Multiplication by $C\gamma^0$;

3) Change of $A^\mu \to -A^\mu$.

The physical meaning of charge conjugation is that for any physically realizable state of electron in the field A^μ corresponds a physically realizable state of a positron in the field $-A^\mu$. Thus, the operation of charge conjugation changes electrons with positive energy and spin up on positrons with positive energy and spin down.

4.6 Quantization

The Dirac equation can be obtained by using the variational principle, starting from the following *Lorentz* invariant action[6]

$$S = \int \mathrm{d}x \Big[\frac{i}{2}\Big(\bar\psi\gamma^\mu\partial_\mu\psi - \partial_\mu\bar\psi\gamma^\mu\psi\Big) - \frac{mc}{\hbar}\bar\psi\psi\Big] = c\int \mathrm{d}\vec{x}\,\mathrm{d}t\,\bar\psi\Big(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\Big)\psi\,. \tag{4.46}$$

We remind that in the action we have $\mathrm{d}x = c\,\mathrm{d}\vec{x}\mathrm{d}t$. Here ψ and $\bar\psi$ can be considered as independent (spinor) variables. With such normalization of the action, the latter is assumed to be dimensionless, the fermions have dimension $[\ell^{-3/2}]$, where ℓ is the length. The Lagrangian density is therefore

$$\mathscr{L} = c\,\bar\psi\Big(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\Big)\psi\,.$$

Note that the quantity $\frac{mc}{\hbar}$ is nothing else but the inverse *the Compton wave length* $\lambda_c = \frac{\hbar}{mc}$ of the Dirac field.

The canonical momentum is

$$\pi = \frac{\delta S}{\delta\dot\psi} = i\bar\psi\gamma^0 = i\psi^{*t}\,. \tag{4.47}$$

As we see, the canonical momentum does not depend on the velocity $\dot\psi$ at all, and therefore the Dirac Lagrangian falls in a category of singular dynamical systems. To obtain the proper Hamiltonian description of the system one has to use the so-called Dirac bracket formalism. Here we use however a simpler alternative approach based on the consideration of equations of motion. First we note that the Hamiltonian can be obtained through the standard formula

$$H = \int \mathrm{d}\vec{x}\big(\pi\dot\psi - \mathscr{L}\big) = \frac{1}{\hbar}\int \mathrm{d}\vec{x}\Big(-i\hbar\,c\,\bar\psi\gamma^i\partial_i\psi + mc^2\,\bar\psi\psi\Big) = \frac{1}{\hbar}\int \mathrm{d}\vec{x}\,\psi^\dagger\Big(\underbrace{c\alpha^i(-i\hbar\partial_i) + mc^2\,\beta}_{\mathscr{H}}\Big)\psi\,,$$

where we recall that $\mathscr{H}$ is the Hamiltonian operator of the first quantized theory. One can see that $[H] = [E/\hbar] = [1/t]$.

[6]The sign of the Lagrangian as well as an overall coefficient can be chosen arbitrarily. Since the Lagrangian involve derivatives of ψ and $\bar\psi$ linearly, the corresponding action cannot have neither maximum nor minimum. The condition $\delta S = 0$ defines a stationary point but not an extremum of the integral.

To establish the Poisson structure, we rewrite the Dirac equations in the form of the evolution equations, which must be nothing else but the Hamiltonian equations of motion

$$\begin{aligned}\tfrac{\partial\psi}{\partial t} &= -\tfrac{i}{\hbar}\Big(c\alpha^i(-i\hbar\partial_i) + mc^2\,\beta\Big)\psi = \{H,\psi\}\,,\\ \tfrac{\partial\psi^*}{\partial t} &= +\tfrac{i}{\hbar}\Big(c\alpha^{i*}(i\hbar\partial_i) + mc^2\,\beta\Big)\psi^* = \{H,\psi^*\}\,.\end{aligned} \tag{4.48}$$

These equations of motion follow from the Hamiltonian

$$H = \frac{1}{\hbar}\int \mathrm{d}\vec{x}\,\psi^\dagger(\vec{x})\mathscr{H}(\vec{x})\psi(\vec{x}) \tag{4.49}$$

and the following Poisson bracket

$$\begin{aligned}\{\psi_i(\vec{x}),\psi_j^*(\vec{x}')\} &= i\delta^{(3)}(\vec{x}-\vec{x}')\delta_{ij}\,,\\ \{\psi_i(\vec{x}),\psi_j(\vec{x}')\} &= 0\,,\\ \{\psi_i^*(\vec{x}),\psi_j^*(\vec{x}')\} &= 0\,.\end{aligned} \tag{4.50}$$

where $\psi(\vec{x})$ and $\psi_i^*(\vec{x})$ are considered as the usual classical (commuting) variables. Up to now our considerations were purely classical. In particular, $\psi(\vec{x})$ was treated as a classical four component field satisfying the Dirac equations of motion. The reader might be very much confused by the appearance of $\hbar$ in this classical equations of motion. Do not be confused! The Planck constant is standing there merely as a parameter to provide the proper matching of dimensions of various quantities entering the Dirac equation. Quantization of the Dirac field has not been performed so far.

Now we are in a position to perform quantization of the Dirac field. The fundamental feature of the Dirac field is that it is quantized by means of anti-commutator instead of commutator. This is dictated by the Pauli-Lüders theorem on the relationship between spin and statistics and also by the requirement by positivity of the second quantized Hamiltonian. Thus, our quantization procedure will consist in replacing the classical Poisson brackets with quantum Poisson brackets (this is as usual), while the quantum Poisson brackets will be realized as *anti-commutators*, rather than commutators.

$$\begin{aligned}\{\psi_i(\vec{x}),\psi_j^*(\vec{x}')\}_\hbar &= \frac{i}{\hbar}(\psi_i(\vec{x})\psi_j^*(\vec{x}') + \psi_j^*(\vec{x}')\psi_i(\vec{x})) = i\delta^{(3)}(\vec{x}-\vec{x}')\delta_{ij}\,,\\ \{\psi_i(\vec{x}),\psi_j(\vec{x}')\}_\hbar &= \frac{i}{\hbar}(\psi_i(\vec{x})\psi_j(\vec{x}') + \psi_j(\vec{x}')\psi_i(\vec{x})) = 0\,,\\ \{\psi_i^*(\vec{x}),\psi_j^*(\vec{x}')\}_\hbar &= \frac{i}{\hbar}(\psi_i^*(\vec{x})\psi_j^*(\vec{x}') + \psi_j^*(\vec{x}')\psi_i^*(\vec{x})) = 0\,.\end{aligned} \tag{4.51}$$

From now on we reserve the notation $\{.\,,.\}$ for the *anti-commutator* of fields! In terms of anti-commutators the commutation relations between the components of the quantized Dirac field look like

$$\begin{aligned}\{\psi_i(\vec{x}),\psi_j^\dagger(\vec{x}')\} &= \hbar\,\delta^{(3)}(\vec{x}-\vec{x}')\delta_{ij}\\ \{\psi_i(\vec{x}),\psi_j(\vec{x}')\} &= 0\\ \{\psi_i^\dagger(\vec{x}),\psi_j^\dagger(\vec{x}')\} &= 0\end{aligned} \tag{4.52}$$

Every component $\psi_i(x)$ is now understood as an operator-valued distribution, satisfying the anti-commutation relations above, as well as the following conjugation rules

$$\begin{aligned}(\lambda\psi)^\dagger &= \bar{\lambda}\psi^\dagger\,, \qquad \lambda\in\mathbb{C}\,,\\ (\psi_i\psi_j)^\dagger &= \psi_j^\dagger\psi_i^\dagger\,,\\ \psi^{\dagger\dagger} &= \psi\,.\end{aligned} \tag{4.53}$$

Recall that we obtained

$$\psi(x) = \int \frac{\mathrm{d}\vec{p}}{(2\pi\hbar)^{3/2}} \left(\frac{mc^2}{E(p)}\right)^{1/2} \sum_{r=1}^{2} \left(b_r(p) u_+^r(p) e^{-\frac{i}{\hbar} p_\mu x^\mu} + d_r^\dagger(p) v_-^r(p) e^{\frac{i}{\hbar} p_\mu x^\mu} \right). \tag{4.54}$$

Introduce the following anti-commutation relations

$$\begin{aligned} &\{b_r(\vec{p}), b_s^\dagger(\vec{p}')\} = \{d_r(\vec{p}), d_s^\dagger(\vec{p}')\} = \hbar\delta_{rs}\delta(\vec{p}-\vec{p}')\,, \\ &\{b_r(\vec{p}), b_s(\vec{p}')\} = \{d_r(\vec{p}), d_s(\vec{p}')\} = 0\,, \\ &\{b_r(\vec{p}), d_s(\vec{p}')\} = \{b_r(\vec{p}), d_s^\dagger(\vec{p}')\} = \{b_r^\dagger(\vec{p}), d_s(\vec{p}')\} = 0\,. \end{aligned} \tag{4.55}$$

Compute the equal-time anti-commutator[7]

$$\begin{aligned} \{\psi_i(\vec{x}), \psi_j^\dagger(\vec{x}')\} &= \int \frac{\mathrm{d}p\mathrm{d}p'}{(2\pi\hbar)^3} \frac{mc^2}{(E(p)E(p'))^{1/2}} \times \\ &\times \Big[\sum_{r,s=1}^{2} \{b_r(\vec{p}), b_s^\dagger(\vec{p}')\} (u_+^r(p))_i (u_+^s(p'))_j^* e^{-\frac{i}{\hbar}(p_0-p_0')x^0 - \frac{i}{\hbar}(p_i x^i - p_i' x^{i'})} + \\ &+ \sum_{r,s=1}^{2} \{d_r^\dagger(\vec{p}), d_s(\vec{p}')\} (v_-^r(p))_i (v_-^s(p'))_j^* e^{+\frac{i}{\hbar}(p_0-p_0')x^0 + \frac{i}{\hbar}(p_i x^i - p_i' x^{i'})} \Big]. \end{aligned} \tag{4.56}$$

$$\begin{aligned} \{\psi_i(\vec{x}), \psi_j^\dagger(\vec{x}')\} &= \hbar \int \frac{\mathrm{d}p}{(2\pi\hbar)^3} \frac{mc^2}{E(p)} \times \\ &\times \Big[\sum_{r=1}^{2} (u_+^r(p))_i (u_+^s(p))_j^* \, e^{-\frac{i}{\hbar} p_i (x^i - x^{i'})} + \sum_{r=1}^{2} (v_-^r(p))_i (v_-^r(p))_j^* \, e^{\frac{i}{\hbar} p_i (x^i - x^{i'})} \Big]. \end{aligned} \tag{4.57}$$

From the relations (4.36) we find

$$\begin{aligned} (u_+^r(p))_i (u_+^s(p))_j^* &= (u_+^r(p))_i (\bar{u}_+^s(p)\gamma^0)_j = (\Lambda_+(p)\gamma^0)_{ij} = \frac{\Big(E(p)\mathbb{1} + c\alpha^i p^i + mc^2\beta\Big)_{ij}}{2mc^2}\,, \\ (v_-^r(p))_i (v_-^s(p))_j^* &= (v_-^r(p))_i (\bar{v}_-^s(p)\gamma^0)_j = -(\Lambda_-(p)\gamma^0)_{ij} = \frac{\Big(E(p)\mathbb{1} + c\alpha^i p^i - mc^2\beta\Big)_{ij}}{2mc^2}\,. \end{aligned} \tag{4.58}$$

Thus, we arrive at

$$\begin{aligned} \{\psi_i(\vec{x}), \psi_j^\dagger(\vec{x}')\} &= \hbar \int \frac{\mathrm{d}p}{(2\pi\hbar)^3} \frac{mc^2}{E(p)} \times \\ &\times \left[\frac{\Big(E(p)\mathbb{1} + c\alpha^i p^i + mc^2\beta\Big)_{ij}}{2mc^2} \, e^{-\frac{i}{\hbar} p_i (x^i - x^{i'})} + \frac{\Big(E(p)\mathbb{1} + c\alpha^i p^i - mc^2\beta\Big)_{ij}}{2mc^2} \, e^{\frac{i}{\hbar} p_i (x^i - x^{i'})} \right]. \end{aligned}$$

Making in the second term the change of variables $\vec{p} \to -\vec{p}$, we obtain

$$\{\psi_i(\vec{x}), \psi_j^\dagger(\vec{x}')\} = \hbar \int \frac{\mathrm{d}p}{(2\pi\hbar)^3} \frac{mc^2}{E(p)} \frac{2E(p)}{2mc^2} \, e^{-\frac{i}{\hbar} p_i (x^i - x^{i'})} = \int \frac{\mathrm{d}p}{(2\pi\hbar)^3} \, e^{-\frac{i}{\hbar} p_i (x^i - x^{i'})} \, \delta_{ij} = \hbar\delta(\vec{x}-\vec{x}')\delta_{ij}$$

Acting on $\psi(x)$ with the Hamiltonian $\mathscr{H} = c\alpha^i(-i\hbar\partial_i) + mc^2\,\beta$, we get

$$\mathscr{H}\psi(x) = \int \frac{\mathrm{d}\vec{p}}{(2\pi\hbar)^{3/2}} \left(\frac{mc^2}{E(p)}\right)^{1/2} E(p) \sum_{r=1}^{2} \left(b_r(p) u_+^r(p) e^{-\frac{i}{\hbar} p_\mu x^\mu} - d_r^\dagger(p) v_-^r(p) e^{\frac{i}{\hbar} p_\mu x^\mu} \right).$$

[7]The time $x^0 = ct$ is taken equal for both $\psi(x)$ and $\psi^\dagger(x)$.

The for the Hamiltonian we get

$$\begin{aligned} H = \int \mathrm{d}\vec{x}\, \psi^\dagger \mathscr{H} \psi \quad &= \quad \int \mathrm{d}\vec{x} \int \frac{\mathrm{d}\vec{p}\mathrm{d}\vec{p}'}{(2\pi\hbar)^3} \frac{mc^2}{(E(p)E(p'))^{1/2}} E(p') \times \\ &\times \quad \sum_{r=1}^{2} \left(b_r^\dagger(p) (u_+^r(p))_i^* e^{\frac{i}{\hbar} p_\mu x^\mu} + d_r(p) (v_-^r(p))_i^* e^{-\frac{i}{\hbar} p_\mu x^\mu} \right) \times \\ &\times \quad \sum_{s=1}^{2} \left(b_s(p') (u_+^s(p'))_i e^{-\frac{i}{\hbar} p'_\mu x^\mu} - d_s^\dagger(p') (v_-^s(p'))_i e^{\frac{i}{\hbar} p'_\mu x^\mu} \right). \end{aligned}$$

Since this computation of one of the most important on the Dirac theory, we do it in a very detailed manner. We have

$$\begin{aligned} H \quad &= \quad \int \mathrm{d}\vec{x}\, \psi^\dagger \mathscr{H} \psi \; = \int \mathrm{d}\vec{x} \int \frac{\mathrm{d}\vec{p}\mathrm{d}\vec{p}'}{(2\pi\hbar)^3} \frac{mc^2}{(E(p)E(p'))^{1/2}} E(p') \times \\ &\times \quad \sum_{r,s=1}^{2} \Big[b_r^\dagger(p) b_s(p') \Big(u_+^{r\dagger}(p) u_+^s(p') \Big) e^{\frac{i}{\hbar}(p_\mu - \vec{p}'_\mu) x^\mu} - b_r^\dagger(p) d_s^\dagger(p') \Big(u_+^{r\dagger}(p) v_-^s(p') \Big) e^{\frac{i}{\hbar}(p_\mu + \vec{p}'_\mu) x^\mu} \\ &+ \quad d_r(p) b_s(p') \Big(v_-^{r\dagger}(p) u_+^s(p') \Big) e^{-\frac{i}{\hbar}(p_\mu + \vec{p}'_\mu) x^\mu} - d_r(p) d_s^\dagger(p') \Big(v_-^{r\dagger}(p) v_-^s(p') \Big) e^{-\frac{i}{\hbar}(p_\mu - \vec{p}'_\mu) x^\mu} \Big]. \end{aligned}$$

Now we can take the integral over $\vec{x}$ which is easy, since it results into delta functions $\delta(\vec{p} - \vec{p}')$ or $\delta(\vec{p} + \vec{p}')$. After this we perform the integration over $\vec{p}'$ and find

$$\begin{aligned} H \quad &= \quad mc^2 \int \mathrm{d}\vec{p} \sum_{r,s=1}^{2} \Big[b_r^\dagger(p) b_s(p) \Big(u_+^{r\dagger}(p) u_+^s(p) \Big) - b_r^\dagger(p) d_s^\dagger(-p) \underbrace{\Big(u_+^{r\dagger}(p) v_-^s(-p) \Big)}_{=0} e^{2\frac{i}{\hbar} p_0 x^0} \\ &\qquad + d_r(p) b_s(-p) \underbrace{\Big(v_-^{r\dagger}(p) u_+^s(-p) \Big)}_{=0} e^{-2\frac{i}{\hbar} p_0 x^0} - d_r(p) d_s^\dagger(p) \Big(v_-^{r\dagger}(p) v_-^s(p) \Big) \Big]. \end{aligned}$$

Note that in the above formulae $\dagger$ applied to spinors u or v means the usual hermition conjugation. Since

$$u_+^{r\dagger}(p) u_+^s(p) = \frac{E(p)}{mc^2} \delta^{rs}, \qquad v_-^{r\dagger}(p) v_-^s(p) = \frac{E(p)}{mc^2} \delta^{rs},$$

we finally find

$$H \quad = \quad \int \mathrm{d}\vec{p}\, E(p) \Big(b_r^\dagger(p) b_r(p) - d_r(p) d_r^\dagger(p) \Big). \tag{4.59}$$

Now we can really see how the problem of negative energies is solved. Since $\{d_r(p), d_s^\dagger(p')\} = \hbar\delta_{rs}\delta(\vec{p} - \vec{p}')$, we can bring the Hamiltonian to the *normal ordered form*

$$H \quad = \quad \int \mathrm{d}\vec{p}\, E(p) \Big(b_r^\dagger(p) b_r(p) + d_r^\dagger(p) d_r(p) - 2\hbar\delta(0) \Big). \tag{4.60}$$

The operators $b_r(p)$ and $b_r^\dagger(p)$ are interpreted as annihilation and creation operators of an electron (fermion) with momentum $\vec{p}$. Analogously, $d_r(p)$ and $d_r^\dagger(p)$ are interpreted as annihilation and creation operators of a positron (anti-fermion) with momentum $\vec{p}$. The infinite contribution to the energy can be throwing away passing to the normal ordered expressions. Actually, such an interpretation of the creation and annihilation operators comes from considering also the second quantized momentum operator and the operator of an electric charge, which in the normal ordered form are

$$\vec{P} = \int \mathrm{d}\vec{p}\, \vec{p} \Big(b_r^\dagger(p) b_r(p) + d_r^\dagger(p) d_r(p) \Big) \tag{4.61}$$

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and

$$Q = \int \mathrm{d}\vec{p} \left(b_r^\dagger(p) b_r(p) - d_r^\dagger(p) d^r(p) \right) . \tag{4.62}$$

4.7 The Dirac propagator

Chapter 5

Electromagnetic field

> Ordinary quantum mechnics cannot give an account of photons which constitute the prime case of relativistic particles. Since photons have the rest mass zero, and correspondingly travel in the vacuum at the velocity, naturally, of light c it is ruled out that a non-relativistic theory such as ordinary quantum mechanics could give even an approximate description.
>
> *Stanford Encyclopedia of Philosophy*

5.1 Classical electromagnetic field

The classical theory of electromagnetism, which is the Maxwell theory, combines together three fundamental and observed phenomena – electricity, magnetism and light. Photons[1] are presented here only in terms of electromagnetic waves which propagate in the vacuum with the speed of light. The theory is relativistic from the very beginning, which is reflected by the covariance of the Maxwell equations under Lorentz transformations. Quantization of the classical electromagnetic field must reveal the particle nature of light.

As is known from the course on classical field theory, the Lagrangian formulation of the Maxwell electromagnetic theory is based on the four-vector electromagnetic potential, which is an underlying field (a Lorentz invariant 4-vector) with the components

$$A^{\mu}=\left(\varphi\left(x\right),\vec{A}\left(x\right)\right),\qquad A_{\mu}=\eta_{\mu\nu}A^{\nu}=\left(\varphi\left(x\right),-\vec{A}\left(x\right)\right). \tag{5.1}$$

Here $\varphi(x)$ and $\vec{A}(x)$ are called the scalar and vector potentials, respectively. The relationship between electric and magnetic fields and the corresponding components of the four-potential are

$$\vec{E}=-\vec{\nabla}\varphi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\qquad\text{and}\qquad\vec{H}=\operatorname{rot}\vec{A}. \tag{5.2}$$

The the action for the classical electrodynamics (without sources) reads as[2]

$$S=-\frac{1}{4c}\int\mathrm{d}^{4}xF_{\mu\nu}F^{\mu\nu}=-\frac{1}{4}\int\mathrm{d}t\,\mathrm{d}^{3}\vec{x}\,F_{\mu\nu}F^{\mu\nu},$$

[1] The name photon has been coined by the chemist Gilbert N. Lewis in 1926.

[2] Normalization of the action $-1/4c$ is written in the the Heaviside system of units; in Gaussian sytem of units it would be $-1/(16\pi)$, as for instance in the Landau and Lifshitz 2nd volume "Classical Field Theory".

where $F_{\mu\nu}$ is the electromagnetic tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{5.3}$$

In terms of electric and magnetic fields the tensor of the electromagnetic field is parametrized as follows (here the index $\mu = 0,1,2,3$ enumerates the rows and the index $\nu = 0,1,2,3$ enumerates columns)[3]

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix}, \qquad F^{\mu\nu} = \eta^{\mu\sigma}\eta^{\nu\rho}F_{\sigma\rho} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix},$$

where we have defined the F_{0i} components to be the electric fields and the F_{ij} components are related to the components of the magnetic fields. In what follows we set

$$\vec{E} = (E_x, E_y, E_z)\,, \qquad \vec{H} = (H_x, H_y, H_z)\,, \qquad \vec{A} \equiv (A_x, A_y, A_z) = (A^1, A^2, A^3)$$

For reader's convenience we also present the relationship between the electromagnetic tensor and its components via indices

$$E_i = F_{0i} = F^{i0} = -F^{0i}, \quad F^{ik} = -\epsilon_{ikl}H_l\,, \qquad H_i = -\frac{1}{2}\epsilon_{ikl}F^{kl}\,.$$

In particular, we stress the relation

$$H_i = -\frac{1}{2}\epsilon_{ikl}F^{kl} = -\frac{1}{2}\epsilon_{ikl}F_{kl} = -\epsilon_{ikl}\partial_k A_l = \epsilon_{ikl}\partial_k A^l = (\mathrm{rot}\,\vec{A})_i\,,$$

that is indeed $\vec{H} = \mathrm{rot}\vec{A}$.

The definition of the electromagnetic tensor (5.3) implies the so-called Bianchi identity

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0 \tag{5.4}$$

and they are equivalent to

$$\vec{\nabla}\cdot\vec{H} = 0\,, \qquad \frac{1}{c}\frac{\partial\vec{H}}{\partial t} = -\vec{\nabla}\times\vec{E}\,. \tag{5.5}$$

In absence of sources equations of motion are

$$\partial_\mu F^{\mu\nu} = 0 \tag{5.6}$$

and they are equivalent to

$$\vec{\nabla}\cdot\vec{E} = 0\,, \qquad \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \vec{\nabla}\times\vec{H}\,. \tag{5.7}$$

[3]The conventions here are that of the the Landau and Lifshitz 2nd volume "Classical Field Theory".

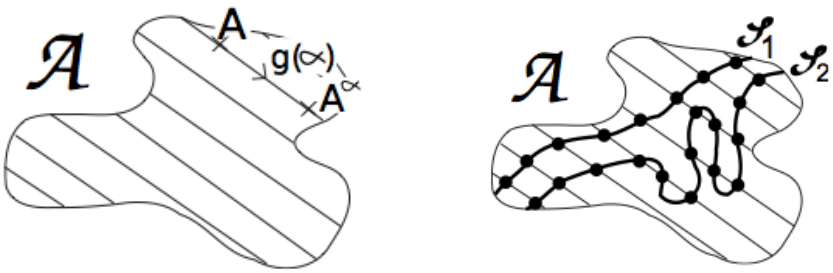

Figure 5.1: The left figure represents the gauge orbits. The right figure shows two gauge fixings – the first one $\mathscr{J}_1$ is complete and the second $\mathscr{J}_2$ is not complete.

5.2 Gauge symmetry

All the physical properties of the electromagnetic field as well as the properties of the charge coupled to the electromagnetic field are determined not by A_μ, but rather by $F_{\mu\nu}$. The underlying reason for this is that electrodynamics exhibits an important new type of symmetry[4]. To understand this issue of symmetry, we may decide to change the vector potential in the following way

$$A_\mu \to A_\mu - \partial_\mu \chi \,, \tag{5.8}$$

which can be rewritten in a less abstract form of space and time components separately:

$$\vec{A} \to \vec{A} + \vec{\nabla}\chi \quad \text{and} \quad \varphi \to \varphi - \frac{1}{c}\frac{\partial \chi}{\partial t} \,. \tag{5.9}$$

These transformations are referred to as *the gauge transformations*. Let us see what effect they have on the tensor of the electromagnetic field:

$$\delta F_{\mu\nu} \;=\; \partial_\mu \left(A_\nu - \partial_\nu \chi\right) - \partial_\nu \left(A_\mu - \partial_\mu \chi\right) - F_{\mu\nu} = -\partial_\mu \partial_\nu \chi + \partial_\nu \partial_\mu \chi = 0 \,.$$

Thus, the Lagrangian as well as the action are invariant under gauge transformations. We know that global symmetries which depend on constant space-time independent parameters lead to conservation laws by the Noether theorem. In opposite, the local gauge symmetries do not lead to conservation laws! A global symmetry takes a physical state and transform it to another physical state, while two states related by a gauge symmetry have to be identified – they correspond to the one and the same physical state. Because of gauge symmetry the Maxwell equations

$$\partial_\mu F^\mu = 0 \quad \Longrightarrow \quad \Big(\eta_{\mu\nu}\partial^\rho \partial_\rho - \partial_\mu \partial_\nu\Big) A^\nu = 0 \tag{5.10}$$

do not have unique solution for all A_μ, because if A_μ solve the equations then $A_\mu + \partial_\mu \alpha$ with $\partial_\mu \alpha\Big|_{t=t_0} = 0$ also do, simply because

$$\Big(\eta_{\mu\nu}\partial^\rho \partial_\rho - \partial_\mu \partial_\nu\Big) \partial^\nu \alpha = 0 \quad \text{for any} \;\; \alpha \,.$$

In other words, since $F_{\mu\nu}$ is the same for both A_μ and $A_\mu + \partial_\mu \alpha$, these configurations correspond to the one and same physical state. Given A_μ the space obtained from A_μ by acting on it with gauge transformations is called the gauge orbit of A_μ, see figure 5.1. Thus, the space of all A_μ is foliated by the gauge orbits. All states belonging to the same orbit describe the same physics and

[4]This symmetry extends to many other physical theories besides electrodynamics.

correspond to the same physical state. One can pick a representative from each gauge orbit – the procedure known as a gauge fixing. Different representative configurations of a physical state are called *different gauges*.

Another peculiarity of the electromagnetic Lagrangian, which is also related to the gauge symmetry, is that equations of motion for A_0 are not dynamical, *i.e.* they do not involve time derivatives of A_0. Indeed,

$$\partial^\mu F_{\mu 0} = 0 \qquad \to \qquad \partial_i(\partial_i A_0 - \partial_0 A_i) = \vec{\nabla}^2 A_0 + \vec{\nabla}\frac{\partial \vec{A}}{\partial t} = 0\,.$$

Thus[5],

$$A_0 = -\vec{\nabla}^{-2}\vec{\nabla}\frac{\partial \vec{A}}{\partial t} = \int \mathrm{d}\vec{x}'\,\frac{\vec{\nabla}\frac{\partial \vec{A}}{\partial t}(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}\,. \tag{5.11}$$

Thus, A_0 is not independent – we do not need to specify it at $t = t_0$ (initial time slice). Thus, the number of independent degrees of freedom cannot be more than three.

Various gauge choices are possible. Below we discuss two of them.

Coulomb (radiation) gauge

$$\vec{\nabla}\cdot\vec{A} = \mathrm{div}\vec{A} = 0 = \partial_i A^i\,. \tag{5.12}$$

For any given $\tilde{A}_i$ one can always find a representative A_i in its gauge orbit which satisfies the Coulomb gauge condition. Indeed,

$$A_i = \tilde{A}_i + \partial_i\alpha\,, \qquad \partial_i A^i = 0\,.$$

We have

$$\partial_i A^i = \partial_i \tilde{A}^i - \vec{\nabla}^2\alpha = 0\,,$$

that is

$$\alpha = \vec{\nabla}^{-2}\partial_i\tilde{A}^i = -\int \mathrm{d}\vec{x}'\,\frac{\vec{\nabla}\cdot\vec{A}(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}\,.$$

From equation (5.11) we deduce that in the Coulomb gauge $A_0 = 0$.

Thus any field configuration A_i can be decomposed into the transverse and longitudinal parts

$$A_i = A_i^\perp + A_i^{||}\,, \tag{5.13}$$

where $\partial_i A_i^\perp = 0$ and $A_i^{||} = \partial_i\xi$ for some ξ. In the Coulomb gauge $\xi = 0$ and only transversal components are present. They represent two physical degrees of freedom corresponding to two possible polarizations of a photon. Obviously, the Coulomb gauge breaks Lorentz symmetry, but this is the simplest gauge to use for quantizing the electromagnetic field.

Lorenz gauge

$$\partial_\mu A^\mu = 0\,. \tag{5.14}$$

This is also an admissible gauge choice as

$$\partial_\mu(\tilde{A}^\mu + \partial^\mu\alpha) = 0 \qquad \to \qquad \partial_\mu\partial^\mu\alpha = -\partial_\mu\tilde{A}^\mu$$

and the last equation (the inhomogeneous wave equation) can also be solved for α. Obviously, the Lorenz gauge does not break Lorentz invariance, but it is an incomplete gauge choice, as it does not pick a unique representative in each gauge orbit, as two configurations A_μ and $A_\mu + \partial_\mu\alpha$ are in the same orbit and satisfy the Lorenz gauge as soon as α is chosen to satisfy $\partial_\mu\partial^\mu\alpha = 0$.

[5]We assume that potentials vanish at infinity so that there is no non-trivial solution of the homogeneous Laplace equation to add.

5.3 Hamiltonian formulation of electrodynamics

To obtain the Hamiltonian formulation of classical electrodynamics (without sources), we start for the Lagrangian for electromagnetic field[6]

$$L = -\frac{1}{4}\int \mathrm{d}\vec{x}\, F_{\mu\nu}F^{\mu\nu}$$

and rewrite it in the so-called first order formalism. To do so, we first compute the canonical momentum conjugate to A^μ. We have

$$p_\mu(x) = \frac{\delta L}{\delta \dot{A}^\mu(x)} = -\int \mathrm{d}^3y\, F^\rho{}_\nu(y)\frac{\delta(\partial_\rho A^\nu(y))}{\delta(\partial_t A^\mu(x))} = -\frac{1}{c}F^0{}_\mu(x) = -\frac{1}{c}F_{0\mu}(x)\,.$$

We see that we have a primary constraint[7]

$$p_0 = 0\,,$$

i.e. the momentum conjugate to A_0 vanishes. This is a straightforward consequence of the fact that the Lagrangian does not contain the time derivative of A_0. In other words, the velocity for A_0 is absent so that A_0 is not a dynamical field! As to the other three components of the canonical momentum, they simply coincide, up to the overall factor $-1/c$, with the electric field:

$$p_i(x) = -\frac{1}{c}F_{0i}(x) = -\frac{1}{c}(\partial_0 A_i - \partial_i A_0) = -\frac{1}{c}E_i\,.$$

This relation allows us to find the velocities $\dot{A}_i$ via the electric field[8]

$$\dot{A}_i = c\,(E_i + \partial_i A_0) \quad \rightarrow \quad \dot{A}^i = -c\,(E_i + \partial_i A_0)\,.$$

Now we write the Lagrangian in the Hamiltonian form

$$L = \underbrace{\int \mathrm{d}^3x\, p_i(x)\dot{A}^i(x)}_{\text{symplectic structure}} - \text{rest}$$

or

$$\text{rest} = \int \mathrm{d}^3x\, p_i(x)\dot{A}^i(x) - L = \int \mathrm{d}^3x\, p_i(x)\dot{A}^i(x) + \frac{1}{4}\int \mathrm{d}^3x\,(-2F_{0i}F_{0i} + F_{ij}F_{ij})\,.$$

The rest must be reexpressed via canonical coordinates and momenta (electric field), *i.e.* all the velocities must be excluded in favor of the canonical momenta. We have

$$\text{rest} = \int \mathrm{d}^3x\, E_i(E_i + \partial_i A_0) + \frac{1}{4}\int \mathrm{d}^3x\,(-2E_i^2 + F_{ij}F_{ij})\,.$$

We also notice that $\vec{H} = \mathrm{rot}\vec{A}$ which can be also written as

$$H_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}\,.$$

Since we have

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\,,$$

[6] We we assume that the action is dimensionless, then $[A_\mu] = \sqrt{c}/\ell$, $[F_{\mu\nu}] = \sqrt{c}/\ell^2$ and $[L] = 1/t$, where ℓ is the length (meters) and t is time (seconds).

[7] Thus, we are dealing with a singular Lagrangian system.

[8] Be careful: $\partial_0 A_i = \frac{1}{c}\dot{A}_i$.

we see that

$$H_i^2 = \frac{1}{4}\epsilon_{ijk}\epsilon_{imn}F_{jk}F_{mn} = \frac{1}{2}F_{ij}F_{ij}\,.$$

Thus, we arrive at

$$\text{rest} = \frac{1}{2}\int \mathrm{d}^3x\left(E_i^2 + H_i^2 - 2A_0\partial_i E_i\right).$$

Thus, the original Lagrangian takes the following form

$$L = -\underbrace{\frac{1}{c}\int \mathrm{d}^3x\, E_i\dot{A}^i}_{\text{symplectic structure}} - \underbrace{\frac{1}{2}\int \mathrm{d}^3x\left(E_i^2 + H_i^2\right)}_{\text{Hamiltonian}} + \underbrace{\int \mathrm{d}^3x\, A_0\partial_i E_i}_{\text{Constraint}}\,.$$

Here

$$\mathrm{H} = \frac{1}{2}\int \mathrm{d}^3x\left(E_i^2 + H_i^2\right)$$

is the Hamiltonian of the electromagnetic field. This is nothing else as the energy of the electromagnetic field! The first term defines the Poisson bracket (recall that $A_i = -A^i$)

$$\{E_i(\vec{x}), A^j(\vec{y})\} = -c\,\delta_{ij}\delta(\vec{x}-\vec{y})\,, \tag{5.15}$$

or

$$\{E_i(\vec{x}), A_j(\vec{y})\} = c\,\delta_{ij}\delta(\vec{x}-\vec{y})\,. \tag{5.16}$$

All the other Poisson brackets vanish. As we see, the right hand side of the Poisson brackets is perfectly compatible with the scaling $[E] = \sqrt{c}/\ell^2$ and $[A] = \sqrt{c}/\ell$.

With these Poisson brackets and the Hamiltonian one can verify the satisfaction of the Hamiltonian equations of motion for $\vec{E}$ and $\vec{H}$.

$$\begin{aligned}\frac{d\vec{E}}{dt} &= \{\mathrm{H}, \vec{E}\} = c\,\vec{\nabla}\times\vec{H}\,,\\ \frac{d\vec{H}}{dt} &= \{\mathrm{H}, \vec{H}\} = -c\,\vec{\nabla}\times\vec{E}\,.\end{aligned}$$

The last term in the Lagrangian contains A_0 which plays the role of the Lagrangian multiplier. Indeed, varying the Lagrangian with respect to A_0 we find the following constraint:

$$C(x) \equiv \partial_i E_i(x) = \mathrm{div}\vec{E} = 0\,,$$

which is nothing else as the Gauss law. As an exercise, one can check that

$$\frac{dC}{dt} = \{\mathrm{H}, C(x)\} = 0\,,$$

that is the constraint is preserved in time. Also, one can easily see that

$$\{C(x), C(y)\} = 0.$$

Some comments are in order.

- We can also verify that the Lagrangian (written in the Hamiltonian form) is invariant with respect to the gauge transformations (see eq.(5.9))

$$\begin{aligned} A^i &\rightarrow A^i + \partial_i \chi \\ A_0 = \varphi &\rightarrow A_0 - \frac{1}{c}\dot{\chi}. \end{aligned}$$

Under the gauge transformations we find

$$\delta L = -\frac{1}{c}\int \mathrm{d}^3x\, E_i \partial_i \dot{\chi} - \frac{1}{c}\int \mathrm{d}^3x\, \dot{\chi}\partial_i E_i = -\frac{1}{c}\int \mathrm{d}^3x\, \partial_i\big(E_i\dot{\chi}\big).$$

Thus, the integrand is the total derivative and we obtain $\delta L = 0$.

- To get the equations of motion for A_i, it is not enough to use the Hamiltonian; one has to take into account the constraint

$$\frac{dA^i(x)}{dt} = \Big\{\mathrm{H} - \int \mathrm{d}^3y\, A_0(y)\partial_i E_i(y), A^i(x)\Big\} = -cE_i(x) - c\partial_i A_0(x)\,.$$

For equations of motion for E_i and H_i adding constraint to the Hamiltonian is possible but not necessary – the constarint commutes with both E_i and H_i and, therefore, gives no contribution to the corresponding equations of motion. The gauge A_0 is called Hamiltonian, because it is in this gauge that equations of motion for all remaining fields are obtained from the Hamiltonian H. This gauge is, however, is not complete – gauge transformations generated by functions $\alpha(\vec{x})$ which do not depend on time preserve the gauge choice $A_0 = 0$.

5.4 Quantization in the Coulomb gauge

Notice that $C(x)$ is actually a generator of gauge transformations. Indeed, we define

$$C = -\frac{1}{c}\int \mathrm{d}\vec{x}\,\alpha(x)\partial_i E_i(\vec{x}) = \frac{1}{c}\int \mathrm{d}\vec{x}\,\partial_i\alpha(x)\,E_i(\vec{x})\,. \tag{5.17}$$

Then we see that

$$\{C, A_i(x)\} = \partial_i\alpha\,. \tag{5.18}$$

Let us write a decomposition for the potential and an electric field into transverse and longitudinal parts

$$A_i = A_i^{\perp} + A_i^{||}\,, \qquad E_i = E_i^{\perp} + E_i^{||} \tag{5.19}$$

or, in a more detailed fashion,

$$A_i = A_i^{\perp} + \partial_i\xi\,, \qquad E_i = E_i^{\perp} + \partial_i\zeta\,, \tag{5.20}$$

where

$$\partial^i A_i^{\perp} = 0\,, \qquad \partial^i E_i^{\perp} = 0\,. \tag{5.21}$$

Decomposition (5.20) is easy to construct. We have

$$\partial_i A_i = \vec{\nabla}^2\xi \quad \rightarrow \quad \xi = \vec{\nabla}^{-2}\partial_j A_j\,.$$

Hence,

$$A_i^{\perp} = A_i - \frac{\partial_i\partial_j}{\vec{\nabla}^2}A_j = \Big(\delta_{ij} - \frac{\partial_i\partial_j}{\vec{\nabla}^2}\Big)A_j\,.$$

The operator

$$P_{ij}^{\perp} = \delta_{ij} - \frac{\partial_i\partial_j}{\vec{\nabla}^2}$$

is a projector on the transverse part of the vector A_i. Indeed,

$$P^{\perp}_{ik}P^{\perp}_{kj} = \left(\delta_{ik} - \frac{\partial_i\partial_k}{\vec{\nabla}^2}\right)\left(\delta_{kj} - \frac{\partial_k\partial_j}{\vec{\nabla}^2}\right) = \delta_{ij} - \frac{\partial_i\partial_j}{\vec{\nabla}^2} - \cancel{\frac{\partial_i\partial_j}{\vec{\nabla}^2}} + \cancel{\frac{\partial_i\partial_j\partial_k\partial_k}{\vec{\nabla}^4}} = P^{\perp}_{ij}\,.$$

Further, note that the field $\vec{H}$ depends on transverse degrees of freedom only

$$H_i = -\epsilon_{ikl}\partial_k A_l = -\epsilon_{ikl}\partial_k A^{\perp}_l - \underbrace{\epsilon_{ikl}\partial_k\partial_l\xi}_{=0}\,.$$

Now we take the Lagrangian and substitute there our decomposition (5.20)

$$L = \underbrace{-\frac{1}{c}\int \mathrm{d}^3x\, E_i\dot{A}^i}_{\text{symplectic structure}} \underbrace{-\frac{1}{2}\int \mathrm{d}^3x\left(E_i^2 + H_i^2\right)}_{\text{Hamiltonian}} + \underbrace{\int \mathrm{d}^3x\, A_0\partial_i E_i}_{\text{Constraint}}\,.$$

We will get

$$L = -\frac{1}{c}\int \mathrm{d}^3x\,(E^{\perp}_i + \partial_i\zeta)(\dot{A}^{i\perp} + \partial^i\dot{\xi}) - \frac{1}{2}\int \mathrm{d}^3x\left((E^{\perp}_i + \partial_i\zeta)^2 + H_i^2\right) + \int \mathrm{d}^3x\, A_0\vec{\nabla}^2\zeta\,.$$

Integrating by parts and using the transversality conditions (5.21), we arrive at

$$L = \underbrace{-\frac{1}{c}\int \mathrm{d}^3x\left(E^{\perp}_i\dot{A}^{i\perp} + \vec{\nabla}^2\zeta\,\dot{\xi}\right)}_{\text{symplectic structure}} \underbrace{-\frac{1}{2}\int \mathrm{d}^3x\left((E^{\perp}_i)^2 - \zeta\vec{\nabla}^2\zeta + H_i^2\right)}_{\text{Hamiltonian}} + \underbrace{\int \mathrm{d}^3x\, A_0\vec{\nabla}^2\zeta}_{\text{Constraint}}\,.$$

Thus, according to this structure of the Lagrangian, the pairs $(E^{\perp}_i, A^{\perp}_i)$ and $(\vec{\nabla}^2\zeta, \xi)$ represent canonically conjugate variables. The Gauss law constraint is then $\vec{\nabla}^2\zeta$, while the Coulomb gauge means $\xi = 0$.[9] As the result, the gauge-fixed Lagrangian involves only transverse degrees of freedom and has the form

$$L = \underbrace{-\frac{1}{c}\int \mathrm{d}^3x\left(E^{\perp}_i\dot{A}^{i\perp}\right)}_{\text{symplectic structure}} \underbrace{-\frac{1}{2}\int \mathrm{d}^3x\left((E^{\perp}_i)^2 + H_i^2\right)}_{\text{Hamiltonian}}\,.$$

The Poisson bracket is

$$\{E^{\perp}_i(\vec{x}), A^{j\perp}(\vec{y})\} = -c\left(\delta_{ij} - \frac{\partial_i\partial_j}{\vec{\nabla}^2}\right)\delta(\vec{x} - \vec{y})\,, \tag{5.22}$$

and it is perfectly compatible with the transversality conditions on $E^{\perp}_i$ and $A^{\perp}_i$.

Quantization can be now performed in a straightforward way, by replacing the Poisson brackets with the quantum Poisson brackets giving

$$[E^{\perp}_i(\vec{x}), A^{j\perp}(\vec{y})] = i\hbar c\left(\delta_{ij} - \frac{\partial_i\partial_j}{\vec{\nabla}^2}\right)\delta(\vec{x} - \vec{y})\,. \tag{5.23}$$

Expanding transverse potentials over plane waves and imposing the commutation relations above, we can interpret, as in the Klein-Gordon theory, the expansion coefficients as creation and annihilation

[9] In the free theory one cannot really see the necessity to impose the gauge condition $\xi = 0$ because imposition of the Gauss constraint alone leaves in the Lagrangian the physical degrees of freedom only. However, in the hamiltonian setting one is allowed to exclude the canonically conjugate variables only by pairs. Moreover, in the interacting theory, where the electromagnetic field interacts with the conserved matter (electron) current, imposition $\xi = 0$ is really necessary, otherwise ξ will be a propagating degree of freedom.

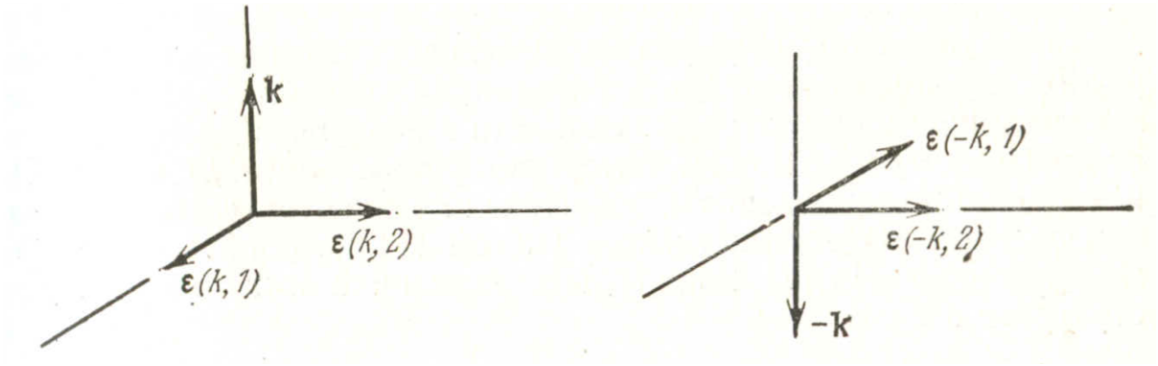

Figure 5.2: Unit polarization vectors of the photon field with momentum $\vec{k}$ and $-\vec{k}$.

operators. The one important feature of the Maxwell theory is that the corresponding quanta carry an integer spin. An expansion over the plane waves has the form

$$\vec{A}^{\perp}(\vec{x},t) = \frac{c}{(2\pi)^{3/2}} \int \frac{\mathrm{d}\vec{k}}{\sqrt{2\omega}} \sum_{r=1}^{2} \vec{\epsilon}_r(\vec{k}) \Big(a^r(\vec{k}) e^{-i\omega t + i\vec{k}\vec{x}} + a^{r\dagger}(\vec{k}) e^{i\omega t - i\vec{k}\vec{x}} \Big) \,. \tag{5.24}$$

For a massless photon we have the dispersion relation $k^0 = \frac{\omega}{c} = |\vec{k}|$, where ω is a frequency and $\vec{k}$ is a wave vector. Here we also introduce two three-dimensional vectors $\vec{\epsilon}_1(\vec{k})$ and $\vec{\epsilon}_2(\vec{k})$ which have unit norm and are orthogonal to the vector $\vec{k}$:

$$\vec{\epsilon}_r(\vec{k}) \cdot \vec{k} = 0 \,. \tag{5.25}$$

The last condition guarantees that $\vec{\nabla} \cdot \vec{k} = 0$. It is also convenient to chose these vectors to be orthogonal to each other for any $\vec{k}$

$$\vec{\epsilon}_r(\vec{k}) \cdot \vec{\epsilon}_s(\vec{k}) = \delta_{rs} \,. \tag{5.26}$$

Two polarization vectors $\vec{\epsilon}_{r,s}(\vec{k})$ describe two physical polarization of the photon and they satisfy the following completeness condition

$$\sum_{r=1}^{2} \epsilon_r^i(\vec{k}) \epsilon_r^j(\vec{k}) = \delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2} \,. \tag{5.27}$$

We note that the explicit polarization vectors can be for instance chosen as follows

$$\vec{\epsilon}_1(\vec{k}) = \begin{pmatrix} 0 \\ -\frac{k^3}{\sqrt{(k^2)^2+(k^3)^2}} \\ \frac{k^2}{\sqrt{(k^2)^2+(k^3)^2}} \end{pmatrix}, \qquad \vec{\epsilon}_2(\vec{k}) = \frac{1}{|\vec{k}|} \begin{pmatrix} \sqrt{(k^2)^2+(k^3)^2} \\ -\frac{k^1 k^2}{\sqrt{(k^2)^2+(k^3)^2}} \\ -\frac{k^1 k^3}{\sqrt{(k^2)^2+(k^3)^2}} \end{pmatrix}. \tag{5.28}$$

The vectors are chosen such as $\vec{\epsilon}_1$ reverse its orientation, while $\vec{\epsilon}_2$ stays the same as $\vec{k} \to -\vec{k}$, see figure 5.2. This implies in particular that

$$\vec{\epsilon}_r(\vec{k}) \cdot \vec{\epsilon}_s(-\vec{k}) = (-1)^r \delta_{rs} \,. \tag{5.29}$$

Since $E_i = -\frac{1}{c}\dot{A}^i$, we find an analogous expansion for the electric field has the form

$$\vec{E}^{\perp}(\vec{x},t) = \frac{i}{(2\pi)^{3/2}} \int \mathrm{d}\vec{k} \sqrt{\frac{\omega}{2}} \sum_{r=1}^{2} \vec{\epsilon}_r(\vec{k}) \Big(a^r(\vec{k}) e^{-i\omega t + i\vec{k}\vec{x}} - a^{r\dagger}(\vec{k}) e^{i\omega t - i\vec{k}\vec{x}} \Big) \,. \tag{5.30}$$

Such an expansion of the transverse potential and the electric field together with their commutators imply the following commutators for the expansion coefficients

$$[a^r(\vec{k}), a^{s\dagger}(\vec{k}')] = \hbar\,\delta^{rs}\delta(\vec{k}-\vec{k}')\,, \tag{5.31}$$

$$[a^r(\vec{k}), a^s(\vec{k}')] = [a^{r\dagger}(\vec{k}), a^{s\dagger}(\vec{k}')] = 0\,. \tag{5.32}$$

Thus, the expansion coefficients acquire the meaning of the creation and annihilation operators corresponding to the two possible polarizations of the photon. It remains to compute the Hamiltonian. Fist we find the contribution of the magnetic field. We have

$$\int \mathrm{d}\vec{x}\, H_i^2 = \int \mathrm{d}\vec{x}\left(\partial_k A^{l\perp}\partial_k A^{l\perp} - \partial_k A^{l\perp}\partial_l A^{k\perp}\right) = \int \mathrm{d}\vec{x}\,\partial_k A^{l\perp}\partial_k A^{l\perp}\,,$$

since the second term in the integrand vanishes upon integrating by parts. With the help of the formulae (5.26) and (5.29) we then find

$$:\int \mathrm{d}\vec{x}\, H_i^2 := \frac{1}{2}\int \mathrm{d}\vec{k}\,\frac{c^2|\vec{k}|^2}{\omega}\sum_{r=1}^{2}\Big[-a^r(\vec{k})a^r(-\vec{k}) + 2a^{r\dagger}(\vec{k})a^r(\vec{k}) - a^{r\dagger}(\vec{k})a^{r\dagger}(\vec{k})\Big]\,.$$

Computation of the contribution of the electric field is elementary

$$:\int \mathrm{d}\vec{x}\,(E_i^{\perp})^2 := \frac{1}{2}\int \mathrm{d}\vec{k}\,\omega\sum_{r=1}^{2}\Big[a^r(\vec{k})a^r(-\vec{k}) + 2a^{r\dagger}(\vec{k})a^r(\vec{k}) + a^{r\dagger}(\vec{k})a^{r\dagger}(\vec{k})\Big]\,.$$

Adding these expressions up and recalling that $\omega = c|\vec{k}|$, we finally arrive at

$$\mathrm{H} = \int \mathrm{d}\vec{k}\,\omega(\vec{k})\sum_{r=1}^{2} a^{r\dagger}(\vec{k})a^r(\vec{k}) = c\int \mathrm{d}\vec{k}\sum_{r=1}^{2}|\vec{k}|\,a^{r\dagger}(\vec{k})a^r(\vec{k})\,. \tag{5.33}$$

This is the Hamiltonian of the electromagnetic field in the Coulomb gauge written in the normal-ordered form via creation and annihilation operators. Analogously one can derive the operator of total momentum

$$\vec{P} = \int \mathrm{d}\vec{x}\,:\vec{E}^{\perp}\times\vec{H} := c\int \mathrm{d}\vec{k}\sum_{r=1}^{2}\vec{k}\,a^{r\dagger}(\vec{k})a^r(\vec{k})\,. \tag{5.34}$$

Some comments are in order.

- Although we constructed the quantum theory of the free electromagnetic field in the Fock space, we performed the quantization *only once* (that is we do not need the second quantization). This makes a difference with the Klein-Gordon and Dirac fields, both involving $\hbar$ already in the first quantized theory, which are attempted to be treated as the Schrödinger equation for a single particle. The Maxwell equations do not involve $\hbar$ at all – they are truly classical fields.
- It is clear from considering the physical Coulomb gauge that the electromagnetic field carries only two degrees of freedom – the field $A_0 = 0$ and A_i fluctuate only in the two-dimensional plane which is orthogonal to the direction of motion, giving rise to two polarizations of a photon.
- We point out that in the formulae (5.33) and (5.34), $\vec{k}$ is not the particle momentum, but rather the wave vector. The particle energy (dispersion) E is related to the frequency, and the wave number is related to the particle momentum $\vec{p}$ by the Planck-Einstein-de Broglie formulae
$$E = \hbar\omega\,, \qquad \vec{p} = \hbar\vec{k}\,,$$
which are valid for both massive particles and light.
- In the Coulomb gauge we have only physical degrees of freedom but the Lorentz invariance is not manifest. One can nevertheless show that the theory is Lorentz invariant by explicitly constructing the generators of the Lorentz algebra in terms of $a^r(\vec{k})$ and $a^{r\dagger}(\vec{k})$ and checking that they indeed form the Lorentz algebra. More severe problems appear in the interacting theory, in particular, in the action there appears a term which is non-local in space. These complications are avoided in the Lorentz gauge. However, in the Lorentz gauge one has unphysical excitations A_0 and $A^{||}$ which has to be quantized but should not change the physical content of the theory. One of the modern ways is to quantize the theory with constraints is to use the so-called Becchi-Rouet-Stora-Tyutin (BRST) quantization.

Make a connection to the classical theory of polarized light.

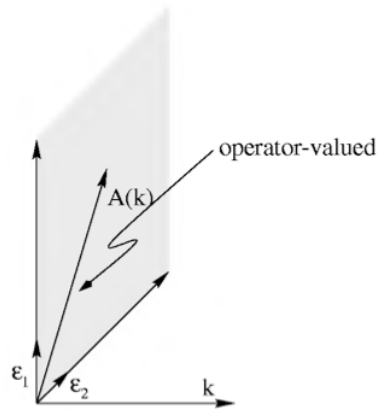

Figure 5.3: The electromagnetic field carries two physical degrees of freedom – the potentials fluctuate in the two-dimensional plane which is orthogonal to the momentum.

5.5 Spin of a photon

Since the action of the electromagnetic field is invariant under Lorentz transformations, then in accord with the Noether theorem, there must be corresponding conserved charges forming the Lorentz algebra. In the case of the electromagnetic field the rotations of the three-dimensional space are represented by

$$\Lambda^{ij} = \int \mathrm{d}\vec{x} \Big[\underbrace{: \dot{A}^k \Big(x^i \frac{\partial}{\partial x_j} - x^j \frac{\partial}{\partial x_i} \Big) A^k :}_{\text{orbital part}} - \underbrace{: (\dot{A}^i A^j - \dot{A}^j A^i) :}_{\text{spin part}} \Big] . \tag{5.35}$$

For the electromagnetic field the spin part

$$S^{ij} = \int \mathrm{d}\vec{x} \, : (\dot{A}^i A^j - \dot{A}^j A^i) :$$

is preserved by itself because

$$\frac{dS^{ij}}{dt} = \int \mathrm{d}\vec{x} \, : (\ddot{A}^i A^j - \ddot{A}^j A^i) := c^2 \int \mathrm{d}\vec{x} \, : (\partial_k^2 A^i A^j - \partial_k^2 A^j A^i) := 0 \, , \tag{5.36}$$

where we have used the fact that A^i solves the wave equation $\Box A^i = 0$ and, in the last step, the integration by parts. In the Coulomb gauge we compute

$$S_i = \frac{1}{2c^2} \epsilon_{ijk} S^{jk} = \frac{1}{c^2} \int \mathrm{d}\vec{x} \, : \epsilon_{ijk} \, \dot{A}^{\perp j} A^{\perp k} : \, .$$

The careful computation of this quantity gives

$$\begin{aligned} S_i = -\frac{i}{2} \int \mathrm{d}\vec{k} \Big[& \epsilon_{imn} \epsilon_r^m(\vec{k}) \epsilon_s^n(-\vec{k}) \, a^r(\vec{k}) a^s(-\vec{k}) e^{-2i\omega t} + \epsilon_{imn} \epsilon_r^m(\vec{k}) \epsilon_s^n(\vec{k}) \, a^{s\dagger}(\vec{k}) a^r(\vec{k}) \\ & - \epsilon_{imn} \epsilon_r^m(\vec{k}) \epsilon_s^n(\vec{k}) \, a^{r\dagger}(\vec{k}) a^s(\vec{k}) - \epsilon_{imn} \epsilon_r^m(\vec{k}) \epsilon_s^n(-\vec{k}) \, a^{r\dagger}(\vec{k}) a^{s\dagger}(-\vec{k}) e^{2i\omega t} \Big] . \end{aligned}$$

Now, with the help of the following identities

$$\epsilon_{imn} \epsilon_r^m(\vec{k}) \epsilon_s^n(\vec{k}) = \frac{k^i}{|\vec{k}|} \epsilon_{rs} \, , \qquad \epsilon_{imn} \epsilon_r^m(\vec{k}) \epsilon_s^n(-\vec{k}) = \frac{k^i}{|\vec{k}|} (1 - \delta_{rs}) \, ,$$

Figure 5.4: The left picture represents a right-polarized photon with the helicity +1, while in the right picture a photon is left-polarized with the helicity −1.

it is easy to find

$$S_i = i \int \mathrm{d}\vec{k}\, \frac{k^i}{|\vec{k}|} \Big(a^{1\dagger}(\vec{k}) a^2(\vec{k}) - a^{2\dagger}(\vec{k}) a^1(\vec{k}) \Big) \,. \tag{5.37}$$

With this expression at hand, one can very by direct computation that $[\mathrm{H}, S_i] = 0$, that is S_i are indeed the conserved quantities for any $i = 1, 2, 3$. Obviously, the expression for S_i can be written in the form

$$\vec{S} = i \int \mathrm{d}\vec{k}\, \vec{\epsilon}_3(\vec{k}) \Big(a^{1\dagger}(\vec{k}) a^2(\vec{k}) - a^{2\dagger}(\vec{k}) a^1(\vec{k}) \Big) \,, \tag{5.38}$$

where we have introduced the third vector $\vec{\epsilon}_3(\vec{k})$ with components

$$\epsilon_3^i = \frac{k^i}{|\vec{k}|} \,, \tag{5.39}$$

which is orthogonal to polarization vectors $\vec{\epsilon}_1(\vec{k})$ and $\vec{\epsilon}_2(\vec{k})$. Thus, as is clear from this momentum space expression spin always aligns the direction of motion of a photon. Consider now one-photon states

$$a^{r\dagger}(\vec{k})|0\rangle$$

and act on it with $\vec{S}$. We will get

$$\begin{aligned} \vec{S}\, a^{1\dagger}(\vec{k})|0\rangle &= -i\hbar \vec{\epsilon}_3(\vec{k}) a^{2\dagger}(\vec{k})|0\rangle \,, \\ \vec{S}\, a^{2\dagger}(\vec{k})|0\rangle &= i\hbar \vec{\epsilon}_3(\vec{k}) a^{1\dagger}(\vec{k})|0\rangle \,. \end{aligned}$$

Omitting the notation of the vacuum this can be written in the matrix form

$$\vec{S} \begin{pmatrix} a^{1\dagger}(\vec{k}) \\ a^{2\dagger}(\vec{k}) \end{pmatrix} = \hbar\, \vec{\epsilon}_3(\vec{k}) \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\text{Pauli } \sigma^2} \begin{pmatrix} a^{1\dagger}(\vec{k}) \\ a^{2\dagger}(\vec{k}) \end{pmatrix} . \tag{5.40}$$

The matrix σ^2 has eigenvalues ± 1 and

$$Q = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \quad \rightarrow \quad Q\sigma^2 Q^{-1} = \mathrm{diag}(1, -1) \,. \tag{5.41}$$

Therefore, making the linear combinations

$$\begin{aligned} a_R^\dagger &= \frac{1}{\sqrt{2}} (a^{1\dagger} - i a^{2\dagger}) \,, \\ a_L^\dagger &= \frac{1}{\sqrt{2}} (a^{1\dagger} + i a^{2\dagger}) \,, \end{aligned}$$

which describe the left and right-polarised waves, we have

$$[\mathscr{S}, a_R^\dagger] = \hbar\, a_R^\dagger\,, \quad [\mathscr{S}, a_L^\dagger] = -\hbar\, a_L^\dagger\,.$$

where $\mathscr{S} = \vec{S}\cdot\vec{\epsilon}_3(\vec{k})$ is the helicity operator. Hence, in units of $\hbar$ the projection of spin on the direction of motion is +1 for the right-polarized photon, and it is -1 for the left-polarized one.

Finally, it is not difficult to find the Feynman propagator in the Coulomb gauge

$$D^\perp_{Fij}(x-x') = \langle 0|A_i(x)A_j(x')|0\rangle\,,$$

which can be explicitly written as the four-dimensional integral in Fourier space

$$D^\perp_{Fij}(x-x') = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{e^{-ikx}}{k^2+i\epsilon}\left(\delta_{ij} - \frac{k_ik_j}{|\vec{k}|^2}\right)\,. \tag{5.42}$$

Obviously this propagator satisfies the transversality condition.

Chapter 6

Path integral in quantum mechanics

> One cannot fail to observe that Feynman's principle in particular – and there is no hyperbole – expresses the laws of quantum mechanics in an exemplary neat and elegant manner, notwithstanding the fact that it employs somewhat unconventional mathematics.
>
> *Yourgrau and Mandelstam*

6.1 Gaussian Integrals

Gaussian integrals play an important role in many areas – in probability theory, in quantum mechanics, in quantum field theory, in the theory of phase transitions in statistical physics. Thus, before introducing the functional integral, in this chapter we recall several useful mathematical results concerning gaussian integrals and also properties of gaussian averages.

Consider a positive-definite measure $\rho(x_1, x_2, \ldots, x_n)$ defined on $\mathbb{R}^n$ and properly normalized. The mean value of any function $F(x_1, x_2, \ldots, x_n)$ is defined as

$$\ll F \gg = \int \mathrm{d}^n x F(\vec{x})\rho(\vec{x})\,. \tag{6.1}$$

Consider now a gaussian integral

$$Z(A) = \int \mathrm{d}^n x \exp\Big(-\frac{1}{2}\sum_{i,j=1}^{n} x_i A_{ij} x_j\Big)\,. \tag{6.2}$$

This integral is convergent provided the matrix A is a symmetric complex matrix, the real part of which is non-negative (that means that all eigenvalues of $\mathrm{Re}(A)$ are non-negative) and none of the eigenvalues of a_i is equal to zero:

$$\mathrm{Re}(A) \geqslant 0\,, \qquad a_i \neq 0.$$

Under this conditions one can prove that[1]

$$Z(A) = \int \mathrm{d}^n x \exp\Big(-\frac{1}{2}\sum_{i,j=1}^{n} x_i A_{ij} x_j\Big) = (2\pi)^{n/2}(\det A)^{-1/2}\,. \tag{6.3}$$

[1] If a matrix is complex, one has to pay attention to the square root and an overall sign.

Let us recall the prove for real positive matrices. One-dimensional gaussian integral of the general form ($a > 0$) can be easily computed

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, e^{-\frac{1}{2}ax^2+bx} = \sqrt{\tfrac{2\pi}{a}} e^{\frac{b^2}{2a}}\,.$$

Any real symmetric matrix can be diagonalized by an orthogonal transformation

$$A = ODO^t\,,$$

where O is orthogonal $O^tO = \mathbb{1}$ and $D = a_i\delta_{ij}$ is diagonal. We make a change of variables $x_i = O_{ij}y_j$, so that

$$x_iA_{ij}x_j = y_kO_{ik}A_{ij}O_{jm}y_m = y_k(O^tAO)_{km}y_m = a_ky_k^2\,.$$

The Jacobian $J = |\det(O)| = 1$. Thus, the integral gets factorized

$$Z(A) = \prod_{i=1}^{n}\int \mathrm{d}y_i\, e^{-a_iy_i^2} = (2\pi)^{n/2}(a_1a_2\ldots a_n)^{-1/2} = (2\pi)^{n/2}(\det A)^{-1/2}\,.$$

It is also easy to compute the gaussian integral of a general form

$$Z(A,b) = \int \mathrm{d}^nx \exp\Big(-\frac{1}{2}\sum_{i,j=1}^{n} x_iA_{ij}x_j + \sum_{i=1}^{n} b_ix_i\Big)\,. \tag{6.4}$$

To compute this integral one has to find the minimum of the quadratic form

$$\frac{\partial}{\partial x_k}\left(\frac{1}{2}\sum_{i,j=1}^{n} x_iA_{ij}x_j - \sum_{i=1}^{n} b_ix_i\right) = \sum_{j=1}^{n} A_{kj}x_j - b_k = 0$$

or with the help of the inverse matrix $b = A^{-1}x$. Finally we make a change of variables

$$x_i = (A^{-1}b)_i + y_i$$

and the integral takes the form

$$Z(A,b) = \exp\Big[\tfrac{1}{2}\sum_{i,j=1}^{n} b_i(A^{-1})_{ij}b_j\Big]\int \mathrm{d}^ny \exp\Big(-\frac{1}{2}\sum_{i,j=1}^{n} y_iA_{ij}y_j\Big)\,. \tag{6.5}$$

This finally gives

$$Z(A,b) = (2\pi)^{n/2}(\det A)^{-1/2}\exp\Big[\tfrac{1}{2}\sum_{i,j=1}^{n} b_i(A^{-1})_{ij}b_j\Big] \tag{6.6}$$

The characteristic property of a gaussian integral is that it remains gaussian after integrating over one variable. This structural rigidity explains a rigidity of gaussian probability distributions.

6.2 Path integral in quantum mechanics

We start with the *the Schrödinder equation – the main equation of quantum mechanics*

$$i\hbar\frac{d\psi(t)}{dt} = H\psi(t)\,, \tag{6.7}$$

where $\psi(t)$ is a vector in a Hilbert space. Introduce a concept of the evolution operator. The evolution operator $U(t_2, t_1)$ transforms the wave function $\psi(t_1)$ in the Schrödinger representation into the wave function $\psi(t_2)$:

$$\psi(t_2) = U(t_2, t_1)\psi(t_1) = e^{-\frac{i}{\hbar}H(t_2-t_1)}\psi(t_1)\,, \tag{6.8}$$

where H is a Schödinger operator. This formula can be easily understood. The operator

$$e^{\frac{i}{\hbar}Ht_1}\psi(t_1) = e^{\frac{i}{\hbar}Ht_1}e^{-\frac{i}{\hbar}Ht_1}\psi = \psi \tag{6.9}$$

brings the wave to the Heisenberg (time-independent) representation, while $e^{-\frac{i}{\hbar}Ht_2}\psi$ creates again the Schrödinger wave function but at the moment t_2. Obviously, $U(t_2, t_1)$ satisfies Schrödinger equation with respect to t_2. As usual the wave function in coordinate representation is obtained as $\psi(q_1, t) = \langle q_1|\psi(t)\rangle$. Thus, eq.(6.9) leads to

$$\langle q_2|\psi(t_2)\rangle = \int_{-\infty}^{\infty} \mathrm{d}q_1\, \langle q_2|e^{-\frac{i}{\hbar}(t_2-t_1)H}|q_1\rangle\langle q_1|\psi(t_1)\rangle,$$

or

$$\psi(q_2, t_2) = \int_{-\infty}^{\infty} \mathrm{d}q_1\, \langle q_2|e^{-\frac{i}{\hbar}(t_2-t_1)H}|q_1\rangle\, \psi(q_1, t_1)\,. \tag{6.10}$$

This is an integral form of the Schrödinger equation and its kernel is the matrix element of the evolution operator, which we denote as

$$W(q_2, t_1; q_1, t_1) = \langle q_2|e^{-\frac{i}{\hbar}(t_2-t_1)H}|q_1\rangle\,.$$

This kernel obeys two properties

- It is markovian

$$\int \mathrm{d}q_2\, W(q_3, t_3; q_2, t_2)W(q_2, t_2; q_1, t_1) = W(q_3, t_3, q_1, t_1)\,.$$

- It obeys the following normalization $W(q_2, t; q_1, t) = \delta(q_2 - q_1)$.

Now we derive a representation for the matrix element of the evolution kernel in terms of the so-called path integral. To do this, we split the time interval (t_0, t_N) into N interva;s $t_{i+1} - t_i = \Delta$, so that $t_N - t_0 = N\Delta$, and write

$$W(q_N, t_N; q_0, t_0) = \int \mathrm{d}q_{N-1} \ldots \int \mathrm{d}q_1\, W(q_N, t_N; q_{N-1}, t_{N-1}) \ldots W(q_1, t_1; q_0, t_0)\,.$$

We have

$$\begin{aligned} W(q_{i+1}, t_{i+1}; q_i, t_i) &= \langle q_{i+1}|e^{-\frac{i}{\hbar}\Delta}|q_i\rangle = \int \mathrm{d}p_i\, \langle q_{i+1}|p_i\rangle\langle p_i|e^{-\frac{i}{\hbar}\Delta}|q_i\rangle = \\ &= \int \mathrm{d}p_i\, \langle q_{i+1}|p_i\rangle\Big(\langle p_i|q_i\rangle - \frac{i}{\hbar}\Delta\langle p_i|H|q_i\rangle + \ldots\Big)\,. \end{aligned}$$

If the Hamiltonian $H(P, Q)$ ordered such that all P's and on the left from Q's then

$$\langle p_i|H|q_i\rangle = \frac{e^{-\frac{i}{\hbar}p_iq_i}}{\sqrt{2\pi\hbar}}H(p_i, q_i)\,.$$

Thus,

$$W(q_{i+1},t_{i+1};q_i,t_i) = \frac{1}{2\pi\hbar}\int \mathrm{d}p_i\, e^{\frac{i}{\hbar}(q_{i+1}-q_i)p_i}\Big(1-\frac{i}{\hbar}\Delta H(p_i,q_i)+\dots\Big) = \\ = \frac{1}{2\pi\hbar}\int \mathrm{d}p_i\, e^{\frac{i}{\hbar}\Delta\Big[\frac{(q_{i+1}-q_i)p_i}{\Delta}-H(p_i,q_i)\Big]}\,.$$

For the case of a particle moving in a potential $V(Q)$ we have

$$H=\frac{p^2}{2m}+V(Q)$$

and the integral over p_i can be computed explicitly, because it is gaussian. Indeed, we have

$$\frac{(q_{i+1}-q_i)p_i}{\Delta}-\frac{p_i^2}{2m} = -\frac{1}{2m}\Big(p_i-\frac{q_{i+1}-q_i}{\Delta}m\Big)^2+\frac{m}{2}\Big(\frac{q_{i+1}-q_i}{\Delta}\Big)^2\,.$$

Therefore,

$$W(q_{i+1},t_{i+1};q_i,t_i) = \frac{e^{\frac{i}{\hbar}\Delta\Big[\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\Delta}\right)^2-V(q_i)\Big]}}{2\pi\hbar}\int \mathrm{d}p_i\, e^{-i\frac{\Delta}{2m\hbar}p_i^2}\,.$$

The remaining integral over p_i is a well–known Fresnel integral

$$\int_{-\infty}^{\infty} e^{\pm iap^2}\mathrm{d}p=\sqrt{\frac{\pi}{a}}e^{\pm i\frac{\pi}{4}}\,,\qquad a>0\,. \tag{6.11}$$

Thus, we end up with

$$W(q_{i+1},t_{i+1};q_i,t_i) = \Big(\frac{m}{2\pi i\hbar\Delta}\Big)^{\frac{1}{2}} e^{\frac{i}{\hbar}\Delta\Big[\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\Delta}\right)^2-V(q_i)\Big]}\,.$$

In this way we reduced our original matrix element to the following $N-1$ fold integral

$$W(q_N,t_N;q_0,t_0)=\Big(\frac{m}{2\pi i\hbar\Delta}\Big)^{\frac{N}{2}}\int \mathrm{d}q_{N-1}\dots\int \mathrm{d}q_1\, e^{\frac{i}{\hbar}\Delta\sum_{i=0}^{N-1}\Big[\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\Delta}\right)^2-V(q_i)\Big]}\,.$$

Now we take the limit $N\to\infty$, $\Delta\to 0$ such that $N\Delta=t_N-t_0$ remains fixed. Introducing notation for the limiting measure

$$\mathcal{D}q=\lim_{N\to\infty}\Big(\frac{m}{2\pi i\hbar\Delta}\Big)^{\frac{N}{2}}\mathrm{d}q_1\dots\mathrm{d}q_{N-1}\equiv\mathcal{N}[N-1]\mathrm{d}q_1\dots\mathrm{d}q_{N-1}\,, \tag{6.12}$$

we arrive at the path integral representation for the matrix element of the evolution operator

$$W(q_N,t_N;q_0,t_0)=\int_{\substack{q(t_0)=q_0\\ q(t_N)=q_N}}\mathcal{D}q\, e^{\frac{i}{\hbar}S[q(t)]}\,.$$

Normalization factor in the measure involving $N-1$ integrals was defined as $\mathcal{N}[N-1]$, so that with N integrations we would get

$$\mathcal{N}[N]=\Big(\frac{m}{2\pi i\hbar\Delta}\Big)^{\frac{N+1}{2}}$$

6.3 Classical limit

Path integral assumes summation over all trajectories of a particles whether or not they solve the classical equations of motion. However, in the limit $\hbar \to 0$ one expects one expects solutions of the classical equations of motion to provide a dominant contribution into the path integral. Consider

$$W = \int \mathcal{D}q\, e^{\frac{i}{\hbar} S[q(t)]}\,.$$

As $\hbar \to 0$ the integrand is rapidly oscillating and two neighboring half-waves of say $\cos(\frac{1}{\hbar}S)$ encompass almost the same but opposite in sign areas. Sum of these areas is small and, as a result, the whole integral is small, *c.f.* figure 9.1. However, around a stationary point of the action, *i.e.* a trajectory which delivers the extremum $\delta S[q(t)] = 0$, the functional $\exp\left(\frac{i}{\hbar} S[q(t)]\right)$ does not oscillate providing thereby a dominant contribution. In general a method of computing the asymptotics of W in the limit $\hbar \to 0$ is called a *a method of stationary phase*; we refer the reader to the appendix 9.1 for the brief account of this method.

Let us expand the action around a classical trajectory

$$S[q(t)] = S[q_0(t)] + \frac{1}{2}\int \mathrm{d}t_1' \mathrm{d}t_2' \frac{\partial^2 S[q(t)]}{\partial q(t_1')\partial q(t_2')}\Big|_{q=q_0} (q(t_1') - q_0(t_1'))(q(t_2') - q_0(t_2')) + \dots$$

It is conventional to change the variables $q(t) \to q_0(t) + \sqrt{\hbar}\, q(t)$, so that[2]

$$W = e^{\frac{i}{\hbar} S[q_0(t)]} \int \mathcal{D}\tilde{q}\, e^{\frac{i}{2}\int \mathrm{d}t_1' \mathrm{d}t_2' \frac{\partial^2 S[q(t)]}{\partial q(t_1')\partial q(t_2')}\Big|_{q=q_0} q(t_1')q(t_2')} + \dots \tag{6.13}$$

Here integration runs over trajectories $q(t)$ such that $q(t_{1,2}) = 0$. The approximation of W where all the higher order terms in $\hbar$ are suppressed is called a semi-classical (or WKB) approximation. Using discretization approach the pass integral

$$\int \mathcal{D}\tilde{q}\, e^{\frac{i}{2}\int \mathrm{d}t_1' \mathrm{d}t_2' \frac{\partial^2 S[q(t)]}{\partial q(t_1')\partial q(t_2')}\Big|_{q=q_0} q(t_1')q(t_2')} \sim \det\left[\frac{\partial^2 S[q(t)]}{\partial q(t_1')\partial q(t_2')}\Big|_{q=q_0}\right]^{-1/2}$$

can be formally understood as a determinant of the corresponding operator. There exists several ways to compute this object. Below we demonstrate how to compute it for a free particle by passing back to its discretized version.

Consider the action for a free particle $S[q(t)] = \int \mathrm{d}t\, \frac{m\dot{q}^2}{2}$. Compute the first variational derivative

$$\delta S = \int \mathrm{d}t\, m\dot{q}\delta\dot{q} = -\int \mathrm{d}t\, m\ddot{q}(t)\delta q(t)\,,$$

that is

$$\frac{\delta S}{\delta q(t)} = -m\ddot{q}(t)\,.$$

To compute the second variational derivative we write

$$\frac{\delta S}{\delta q(t)} = -m\int \mathrm{d}t'\, \ddot{q}(t')\delta(t'-t) = -m\int \mathrm{d}t'\, q(t')\delta''(t'-t)\,.$$

Thus,

$$\delta\frac{\delta S}{\delta q(t)} = -m\int \mathrm{d}t'\, \delta q(t')\delta''(t'-t)$$

[2]The measure $\mathcal{D}\tilde{q}$ arises from $\mathcal{D}q$ upon rescaling $q \to \sqrt{\hbar}q$.

and, therefore,

$$\frac{\delta^2 S}{\delta q(t)\delta q(t')} = -m\,\delta''(t'-t)\,.$$

Now we can compute

$$\int \mathcal{D}\tilde{q}\, e^{\frac{i}{2}\int \mathrm{d}t_1'\mathrm{d}t_2' \frac{\partial^2 S[q(t)]}{\partial q(t_1')\partial q(t_2')}\Big|_{q=q_0} q(t_1')q(t_2')} = \int \mathcal{D}\tilde{q}\, e^{i\int_{t_1}^{t_2}\mathrm{d}t \frac{m\dot{q}^2}{2}}\,.$$

The result must depend on the times t_1 and t_2 only! We compute this integral by passing to the discrete version

$$\int \mathcal{D}\tilde{q}\, e^{i\int_{t_1}^{t_2}\mathrm{d}t\frac{m\dot{q}^2}{2}} = \lim_{N\to\infty} \mathcal{N}[N]\,\hbar^{N/2}\int \mathrm{d}q_1 \ldots \int \mathrm{d}q_N \exp\Big(\frac{im}{2\Delta}\sum_{i=0}^{N+1}(q_{i+1}-q_i)^2\Big)\,. \tag{6.14}$$

Here $q_i = q(t_i)$ and the initial and final conditions are $q_0 = q(t_1) = 0 = q(t_2) = q_{N+1}$. Then

$$\sum_{i=0}^{N+1}(q_{i+1}-q_i)^2 = \sum_{i,j=1}^{N} q_i A_{ij} q_j\,,$$

where the matrix A is depicted on figure 6.3. Hence, once again we arrive at a factorizable expression

$$\begin{aligned}\int \mathcal{D}q\, e^{i\int_{t_1}^{t_2}\mathrm{d}t\frac{m\dot{q}^2}{2}} &= \lim_{N\to\infty}\mathcal{N}[N]\,\hbar^{N/2}\int \mathrm{d}q_1\ldots\int \mathrm{d}q_N \exp\Big(\frac{im}{2\Delta}\sum_{i,j=1}^{N} q_i A_{ij} q_j\Big) = \\ &= \lim_{N\to\infty}\mathcal{N}[N]\,\hbar^{N/2}\int \mathrm{d}q_1\ldots\int \mathrm{d}q_N \exp\Big(\frac{im}{2\Delta}\sum_{i}^{N} a_i q_i^2\Big)\,. \end{aligned}\tag{6.15}$$

Every individual integral here does converge, and is, in fact, nothing else but again the complete Fresnel integral

$$\int_{-\infty}^{\infty}\mathrm{d}q_i \exp\Big(\frac{i}{2}\frac{m}{\Delta}a_i q_i^2\Big) = 2\sqrt{\frac{\pi\Delta}{2ma_i}}e^{\frac{i\pi}{4}} = \sqrt{\frac{2\pi i\Delta}{ma_i}}\,. \tag{6.16}$$

It is easy to find that $\det A_1 = 2$, $\det A_2 = 3$, which leads to the natural assumption that $\det A_N = N+1$. By expanding the determinat of A_N over last column one gets recurrence relation

$$\det A_N = 2\det A_{N-1} - \det A_{N-2}\,.$$

which is obviously satisfied by $\det A_N = N+1$. This leads to the following result

$$\int \mathcal{D}\tilde{q}\, e^{i\int_{t_1}^{t_2}\mathrm{d}t\frac{m\dot{q}^2}{2}} = \mathcal{N}\Big(\frac{2\pi i\hbar\Delta}{m}\Big)^{N/2}\frac{1}{\sqrt{N+1}}\,. \tag{6.17}$$

Taking into account that

$$\mathcal{N}[N] = \Big(\frac{m}{2\pi i\hbar\Delta}\Big)^{(N+1)/2},$$

we then get

$$\int \mathcal{D}\tilde{q}\, e^{i\int_{t_1}^{t_2}\mathrm{d}t\frac{m\dot{q}^2}{2}} = \lim_{N\to\infty}\Big(\frac{m}{2\pi i\hbar\Delta}\Big)^{(N+1)/2}\Big(\frac{2\pi i\hbar\Delta}{m}\Big)^{N/2}\frac{1}{\sqrt{N+1}} = \sqrt{\frac{m}{2\pi i\hbar\Delta(N+1)}}\,.$$

Restoring all the details the final result reads

$$W(q_2,t_2;q_1,t_1) = \sqrt{\frac{m}{2\pi i\hbar(t_2-t_1)}}\exp\Big[\frac{i}{\hbar}\frac{m}{2}\frac{(q_2-q_1)^2}{(t_2-t_1)}\Big]\,. \tag{6.18}$$

$$\mathsf{A} = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Figure 6.1: The $N \times N$ matrix A arising in evaluation the path integral for a free particle. Matrix A is positive definite, . *i.e.* all its eigenvalues are positive.

For free particle this result is exact, because semi-classical expansion stops at the second functional derivative; all higher functional derivatives of S are identically zero. One can easily verify by explicit calculation that (6.18) solves the Schödinger equation

$$i\hbar \frac{\partial}{\partial t_2} W(q_2, t_2; q_1, t_1) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q_2^2} W(q_2, t_2; q_1, t_1)\,.$$

Note that the original definition (6.8) of the evolution operator is valid for $t_2 \geqslant t_1$ only, which therefore also holds for the kernel $W(q_2, t_2; q_1, t_1)$. It is convenient to require that W vanishes for $t_2 < t_1$. This boundary condition (in time) can be incorporated by considering

$$G(q_2, t_2; q_1, t_1) = \theta(t_2 - t_1) W(q_2, t_2; q_1, t_1) \tag{6.19}$$

Here $\theta(t)$ is the so-called step function defined as

$$\theta(t) = \begin{cases} 1 & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t < 0 \end{cases} \tag{6.20}$$

and it has a property that $\frac{d\theta(t)}{dt} = \delta(t)$. From definition of G we deduce that it obeys the following equation

$$\Big(i\hbar \frac{\partial}{\partial t_2} - H(p_2, q_2)\Big) G(q_2, t_2; q_1, t_1) = i\hbar \delta(t_2 - t_1)\delta(q_2 - q_1)\,, \tag{6.21}$$

i.e. $G(q_2, t_2; q_1, t_1)$ is nothing else but Green's function which solves the Cauchy problem for the Schrödinger equation. By construction this is the so-called *retarded* Green's function.

Of course, for free particle there is a much easier way to get the same result.

$$\begin{aligned} \langle q_2 | e^{-\frac{i}{\hbar}(t_2 - t_1)H} | q_1 \rangle &= \int \mathrm{d}p \langle q_2 | e^{-\frac{i}{\hbar}(t_2 - t_1)H} | p \rangle \langle p | q_1 \rangle = \int \mathrm{d}p \, \langle q_2 | p \rangle \langle p | q_1 \rangle e^{-\frac{i}{\hbar}(t_2 - t_1)\frac{p^2}{2m}} = \\ &= \frac{1}{2\pi\hbar} \int \mathrm{d}p \, e^{\frac{i}{\hbar}(q_2 - q_1)p - \frac{i}{\hbar}(t_2 - t_1)\frac{p^2}{2m}}\,. \end{aligned}$$

The last integral is gaussian and to compute it, we complete the p-dependent term in the exponential to the complete square, getting thereby

$$\begin{aligned} \langle q_2 | e^{-\frac{i}{\hbar}(t_2 - t_1)H} | q_1 \rangle &= \frac{1}{2\pi\hbar} e^{\frac{i}{\hbar}\frac{m}{2}\frac{(q_2 - q_1)^2}{(t_2 - t_1)}} \int \mathrm{d}p \, e^{-\frac{i}{\hbar}\frac{t_2 - t_1}{2m}\left(p - m\frac{q_2 - q_1}{t_2 - t_1}\right)^2} = \\ &= \frac{1}{2\pi\hbar} \sqrt{\frac{2m\hbar\pi}{i(t_2 - t_1)}}\, e^{\frac{i}{\hbar}\frac{m}{2}\frac{(q_2 - q_1)^2}{(t_2 - t_1)}} = \sqrt{\frac{m}{2\pi i\hbar(t_2 - t_1)}}\, e^{\frac{i}{\hbar}\frac{m}{2}\frac{(q_2 - q_1)^2}{(t_2 - t_1)}}\,. \end{aligned}$$

Chapter 7

Functional methods in QFT

> The act of creation may be represented as a source, and that of destruction by a sink, which is, in a manner of speaking, a source.... The vacuum at $t = -\infty$ evolves into the vacuum at $t = +\infty$, via the creation, interaction and destruction of a particle, through the agency of a source. We want to know the vacuum-to-vacuum transition amplitude in the presence of a source. This formulation, using the language of sources, is due to Schwinger (1965).
>
> *Lewis Ryder*
> *Quantum Field Theory*

7.1 Generating functional of Green's functions

Consider a free scalar field $\phi(x)$ with the action (the God-given units are in here!)

$$S[\phi(x)] = \int \mathrm{d}^4x \Big(\tfrac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \tfrac{1}{2}m^2\phi^2(x) \Big) . \tag{7.1}$$

Suppose the scalar field $\phi(x)$ has a source $J(x)$. Then we can define the so-called *vacuum-to-vacuum transition amplitude* in the presence of the source J

$$Z_0[J] = \int \mathscr{D}\phi \, \exp\left\{ i \int \mathrm{d}^4x \Big[\mathscr{L}(\phi) + J(x)\phi(x) + \frac{i}{2}\epsilon\phi^2 \Big] \right\} \tag{7.2}$$

The ϵ-dependent factor with $\epsilon \to 0^+$ provides a convergence of the integral for large ϕ.

Integrating by parts in the Lagrangian we can rewrite the quantity above as

$$Z_0[J] = \int \mathscr{D}\phi \, \exp\left\{ -i \int \mathrm{d}^4x \Big[\tfrac{1}{2}\phi(\Box + m^2 - i\epsilon)\phi - J\phi \Big] \right\} . \tag{7.3}$$

Let us make a shift $\phi \to \phi + \phi_0$. Then we get

$$\int \mathrm{d}^4x \Big[\tfrac{1}{2}\phi(\Box + m^2 - i\epsilon)\phi - J\phi \Big] \to$$
$$\int \mathrm{d}^4x \Big[\tfrac{1}{2}\phi(\Box + m^2 - i\epsilon)\phi + \tfrac{1}{2}\phi_0(\Box + m^2 - i\epsilon)\phi_0 + \phi(\Box + m^2 - i\epsilon)\phi_0 - J\phi - J\phi_0 \Big]$$

and chose ϕ_0 to satisfy

$$(\Box + m^2 - i\epsilon)\phi_0 = J\,.$$

The path integral reduces then

$$Z_0[J] = e^{\frac{i}{2}\int \mathrm{d}^4x J\phi_0} \underbrace{\int \mathscr{D}\phi \, \exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\, \phi(\Box + m^2 - i\epsilon)\phi\right\}}_{N} = N e^{\frac{i}{2}\int \mathrm{d}^4x J\phi_0}\,, \tag{7.4}$$

where N is a normalization prefactor. Since Feynman propagator satisfy the equation

$$(\Box + m^2 - i\epsilon)D_F(x) = -i\delta(x)\,,$$

it can be used to write a solution for ϕ_0

$$\phi_0(x) = i\int \mathrm{d}y\, D_F(x-y)J(y)\,. \tag{7.5}$$

Hence,

$$Z_0(J) = \exp\Big[-\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, J(x)D_F(x-y)J(y)\Big]\,. \tag{7.6}$$

where we have chosen normalization in such a way that $Z_0(0) = 1$. We see that the two-point correlation function, which coincides with the Feynman propagator, can be obtained as

$$D_F(x_1 - x_2) = \langle 0 \mid T(\phi(x_1)\phi(x_2)) \mid 0\rangle = \frac{1}{i^2}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)} Z_0[J]\Big|_{J=0}\,.$$

Similarly, we define the n-point correlation function as

$$G(x_1, x_2, \ldots, x_n) = \langle 0 \mid T(\phi(x_1)\phi(x_2)\ldots\phi(x_n)) \mid 0\rangle = \frac{1}{i^n}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\cdots\frac{\delta}{\delta J(x_n)} Z_0[J]\Big|_{J=0}\,.$$

One can see that all the Green's functions with the odd number of points vanish in the theory of free Klein-Gordon field. The even-point functions are non-trivial, for instance the four-point function is

$$G(x_1, x_2, x_3, x_4) = D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3)\,.$$

It can be easily visualized as a sum of *Feynman diagrams*. Each diagram can be though of as describing a process where particles created at two space-time points, each propagates to one of the other points, and then they are annihilated. In general we have

$$G(x_1, x_2, \ldots x_n) = \sum_{\text{perm}} G(x_{p_1}, x_{p_2})\ldots G(x_{p_{2n-1}}, x_{p_{2n}})\,.$$

This result is known as *Wick's* theorem and it can be alternatively derived by using the field commutation relations.

We finish this section by explicitly mentioning that $Z_0(J)$ is nothing else as the generating functional for Green's functions and it can be represented in the form

$$Z_0(J) = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int \mathrm{d}x_1\ldots\mathrm{d}x_n\, J(x_1)\ldots J(x_n)\, G(x_1, x_2, \ldots, x_n)\,.$$

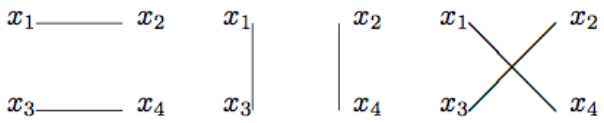

Figure 7.1: Feynman diagrams contributing to the four-point function in the free Klein-Gordon theory.

7.2 Generating functional for interacting fields

Let us now include in the action a potential term $V(\phi)$ which is supposed to be quadratic in the field ϕ:

$$S[\phi(x)] = \int \mathrm{d}^4x \Big(\tfrac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \tfrac{1}{2}m^2\phi^2(x) - V(\phi(x)) \Big) . \tag{7.7}$$

The normalized generating functional is

$$Z[J] = \frac{\int \mathscr{D}\phi \, \exp\Big(iS + i\int J\phi \mathrm{d}x \Big)}{\int \mathscr{D}\phi \, e^{iS}} . \tag{7.8}$$

Obviously, we can write

$$\begin{aligned}
\int \mathscr{D}\phi \, \exp\Big(iS + i\int J\phi \mathrm{d}x \Big) &= \int \mathscr{D}\phi \, \exp\Big\{ -i\int \mathrm{d}x \, V(\phi(x)) \Big\} \exp\Big\{ iS_0 + i\int \mathrm{d}x \, J\phi \Big\} = \\
&= \int \mathscr{D}\phi \, \exp\Big\{ -i\int \mathrm{d}x \, V\Big(\frac{1}{i}\frac{\delta}{\delta J(x)} \Big) \Big\} \exp\Big\{ iS_0 + i\int \mathrm{d}x \, J\phi \Big\} = \\
&= \exp\Big\{ -i\int \mathrm{d}x \, V\Big(\frac{1}{i}\frac{\delta}{\delta J(x)} \Big) \Big\} \int \mathscr{D}\phi \, \exp\Big\{ iS_0 + i\int \mathrm{d}x \, J\phi \Big\} = \\
&= N \exp\Big[-i\int \mathrm{d}x \, V\Big(\frac{1}{i}\frac{\delta}{\delta J(x)} \Big) \Big] \exp\Big[-\frac{1}{2}\int \mathrm{d}x\mathrm{d}y \, J(x) D_F(x-y) J(y) \Big] .
\end{aligned}$$

Analogously,

$$\begin{aligned}
\int \mathscr{D}\phi \, e^{iS} &= \int \mathscr{D}\phi \, \exp\Big(iS + i\int J\phi \mathrm{d}x \Big)\Big|_{J=0} = \\
&= N \exp\Big[-i\int \mathrm{d}x \, V\Big(\frac{1}{i}\frac{\delta}{\delta J(x)} \Big) \Big] \exp\Big[-\frac{1}{2}\int \mathrm{d}x\mathrm{d}y \, J(x) D_F(x-y) J(y) \Big]_{J=0} .
\end{aligned}$$

Thus, for the generating functional the following formula is valid

$$Z[J] = \frac{\exp\Big[-i\int \mathrm{d}z \, V\Big(\frac{1}{i}\frac{\delta}{\delta J(z)} \Big) \Big] \exp\Big[-\frac{1}{2}\int \mathrm{d}x\mathrm{d}y \, J(x) D_F(x-y) J(y) \Big]}{\exp\Big[-i\int \mathrm{d}z \, V\Big(\frac{1}{i}\frac{\delta}{\delta J(z)} \Big) \Big] \exp\Big[-\frac{1}{2}\int \mathrm{d}x\mathrm{d}y \, J(x) D_F(x-y) J(y) \Big]_{J=0}} \tag{7.9}$$

The last formula should be understood in the sense of perturbation theory.

To make further progress, we have to specify the potential. Let us consider an example of the simplest interacting theory in four dimensions where

$$V(\phi) = \frac{g}{4!}\phi^4$$

With this choice of the potential we have

$$Z[J] = \frac{\exp\left[-i\frac{g}{4!}\int \mathrm{d}z\,\frac{\delta^4}{\delta J(z)^4}\right]\exp\left[-\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\,J(x)D_F(x-y)J(y)\right]}{\exp\left[-i\frac{g}{4!}\int \mathrm{d}z\,\frac{\delta^4}{\delta J(z)^4}\right]\exp\left[-\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\,J(x)D_F(x-y)J(y)\right]_{J=0}}\,. \tag{7.10}$$

To only way to treat $Z[J]$ is to expand the interaction term in the power series in g. Taking just the numerator of $Z[J]$ and expanding V up to the first order in g, we will get

$$\left[1 - i\frac{g}{4!}\int \mathrm{d}z\,\frac{\delta^4}{\delta J(z)^4} + \mathcal{O}(g^2)\right]\exp\left[-\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\,J(x)D_F(x-y)J(y)\right] \tag{7.11}$$

Below we evaluate the action of four derivatives on $e^{\Delta} \equiv \exp\left[-\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\,J(x)D_F(x-y)J(y)\right]$ step by step.

1) Action of the first derivative

$$\frac{\delta}{\delta J(z)}e^{\Delta} = -\left[\int D_F(z-y)J(y)\,\mathrm{d}y\right]\,e^{\Delta}\,.$$

2) Action of the second derivative

$$\frac{\delta^2}{\delta J(z)^2}e^{\Delta} = -D_F(0)\,e^{\Delta} + \left[\int D_F(z-y)J(y)\,\mathrm{d}y\right]^2\,e^{\Delta}\,.$$

3) Action of the third derivative

$$\frac{\delta^3}{\delta J(z)^3}e^{\Delta} = 3D_F(0)\left[\int D_F(z-y)J(y)\,\mathrm{d}y\right]\,e^{\Delta} - \left[\int D_F(z-y)J(y)\,\mathrm{d}y\right]^3\,e^{\Delta}\,.$$

4) Action of the fourth derivative

$$\frac{\delta^4}{\delta J(z)^4}e^{\Delta} = 3D_F^2(0)e^{\Delta} - 6D_F(0)\left[\int D_F(z-y)J(y)\,\mathrm{d}y\right]^2\,e^{\Delta} + \left[\int D_F(z-y)J(y)\,\mathrm{d}y\right]^4\,e^{\Delta}\,.$$

We can represent the final result graphically. The propagator is denoted by a straight line, the cross means the source and $D_F(0)$ is a circle

$$\left[1 - i\frac{g}{4!}\int \mathrm{d}z\,\frac{\delta^4}{\delta J(z)^4}\right]e^{\Delta} = \left[1 - i\frac{g}{4!}\int \mathrm{d}z\left[3\,\bigcirc\!\!\bigcirc - 6\times\!\!-\!\!\bigcirc\!\!-\times + \times\right]\right]e^{\Delta}\,. \tag{7.12}$$

The meeting of four lines at a point in all three diagrams is a clearly a consequence of the fact that the interaction term (the potential) is ϕ^4. Moreover, the coefficients 3,6, and 1 in front of the Feynman diagrams follow from simple symmetry considerations. The first term, for instance, results from joining up the two pairs of lines in the third term, in all possible ways; there are three different ways to do this. The second coefficient is obtained by joining any two lines of the third diagram, obviously this can be done in six different ways. These numerical coefficients are known as the *symmetry factors*.

The first term is known as a *vacuum diagram*, because it has no external lines. It is easy to obtain the denominator of $Z[J]$, one has to put $J = 0$ in the terms of the numerator we just obtained. Thus, $Z[J]$ has the form

$$Z[J] = \frac{\left[1 - \frac{ig}{4!}\int \mathrm{d}z\left[3\,\bigcirc\!\!\bigcirc - 6\times\!\!-\!\!\bigcirc\!\!-\times + \times\right] + \ldots\right]e^{\Delta}}{\left[1 - \frac{ig}{4!}\int \mathrm{d}z\left(3\,\bigcirc\!\!\bigcirc\right) + \ldots\right]}\,.$$

In fact, the denominator contains the sum of unity and all vacuum bubbles. If we expand denominator we will get

$$
\begin{aligned}
Z[J] &= \Big[e^{\Delta} - \frac{ig}{4!}\int \mathrm{d}z \Big[3\,\bigcirc\!\!\bigcirc - 6\times\!\!\!-\!\!\bigcirc\!\!-\!\!\!\times + \times\Big]e^{\Delta} + \ldots\Big] \times \Big[1 + \frac{ig}{4!}\int \mathrm{d}z\,\Big(3\,\bigcirc\!\!\bigcirc\Big) + \ldots\Big] = \\
&= e^{\Delta} - \frac{ig}{4!}\int \mathrm{d}z \Big[3\,\bigcirc\!\!\!\!\diagup\!\!\bigcirc - 6\times\!\!\!-\!\!\bigcirc\!\!-\!\!\!\times + \times\Big]e^{\Delta} + \frac{ig}{4!}\int \mathrm{d}z\,\Big(3\,\bigcirc\!\!\!\!\diagup\!\!\bigcirc\Big)e^{\Delta} + \ldots = \\
&= \Big[1 - \frac{ig}{4!}\int \mathrm{d}z\Big[-6\times\!\!\!-\!\!\bigcirc\!\!-\!\!\!\times + \times\Big] + \ldots\Big]e^{\Delta}\,.
\end{aligned}
$$

As we see, the vacuum bubble disappeared from $Z[J]$. Actually, this property will hold to all orders in perturbation theory and it is the general property of normalized generating functionals. Thus, in our further considerations we will just ignore the vacuum diagrams.

Let us compute now two-point Green's function by using our result for $Z[J]$

$$
G(x_1, x_2) = \frac{1}{i^2}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\Big|_{J=0}\,. \tag{7.13}
$$

A simple computation gives

$$
\begin{aligned}
G(x_1, x_2) &= D_F(x_1 - x_2) - \frac{ig}{4!}\,12\,D_F(0)\int \mathrm{d}z\, D_F(z - x_1)D_F(z - x_2) + \ldots = \\
&= D_F(x_1 - x_2) - \frac{ig}{2}\,D_F(0)\int \mathrm{d}z\, D_F(z - x_1)D_F(z - x_2) + \ldots
\end{aligned} \tag{7.14}
$$

We thus clearly see the two-point function in the interacting theory contains a correction to the propagator of the free theory already at the leading order of perturbation theory. This correction has an important physical interpretation which we will now discuss. To this end, we first compute

$$
\begin{aligned}
-\frac{ig}{4!}\,12\,D_F(0)\int \mathrm{d}z\, D_F(z - x_1)D_F(z - x_2) \quad &= \\
= -\frac{ig}{2}\,D_F(0)\int \mathrm{d}^4 z \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}\frac{i}{p^2 - m^2 + i\epsilon}e^{-ik(z-x_1)-ip(z-x_2)} &= \\
= \frac{ig}{2}\,D_F(0)\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{e^{-ik(x_1-x_2)}}{(k^2 - m^2 + i\epsilon)^2}\,. &
\end{aligned}
$$

Hence, we can write

$$
G(x_1, x_2) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i\,e^{-ik(x_1-x_2)}}{k^2 - m^2 + i\epsilon} + \frac{g}{2}\,D_F(0)\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i\,e^{-ik(x_1-x_2)}}{(k^2 - m^2 + i\epsilon)^2} + \ldots\,.
$$

Consider now the expression

$$
\begin{aligned}
\frac{1}{k^2 - m^2 + i\epsilon - \alpha} = \frac{1}{k^2 - m^2 + i\epsilon}\frac{1}{1 - \frac{\alpha}{k^2 - m^2 + i\epsilon}} \approx \frac{1}{k^2 - m^2 + i\epsilon}\Big(1 + \frac{\alpha}{k^2 - m^2 + i\epsilon}\Big) = \\
= \frac{1}{k^2 - m^2 + i\epsilon} + \frac{\alpha}{(k^2 - m^2 + i\epsilon)^2}\,,
\end{aligned}
$$

where we have assumed that α is small. Comparing this expansion with the expression for $G(x_1, x_2)$, we see that the latter can be rewritten as

$$
G(x_1, x_2) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i\,e^{-ik(x_1-x_2)}}{k^2 - m^2 - \frac{g}{2}\,D_F(0) + i\epsilon}\,. \tag{7.15}
$$

We see that the pole of the propagator gets shifted and it equals to

$$m^2 + \frac{g}{2}\, D_F(0) \equiv m^2 + \delta m^2 = m_r^2 \,,$$

where

$$\delta m^2 = \frac{g}{2}\, D_F(0)\,.$$

The mass m_r is called the *physical or renormalized* mass. Notice that

$$D_F(0) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon}$$

is the quadratically divergent quantity. Thus, the original mass m^2 is renormalized by an infinite quantity, but this is in accord with the basic idea of renormalization – a physical quantity (mass, in the present case) is not the same as the parameter in the Lagrangian, if an interaction is present.

Now we look for the four-point function (we now do not take into account the bubles)

$$\begin{aligned}
G(x_1, x_2, x_3, x_4) &= \frac{1}{i^4} \frac{\delta^4 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} = \\
&= \frac{1}{i^4} \frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} \left[1 - \frac{ig}{4!}\int \mathrm{d}z \Big[-6\, \times\!\overset{\bigcirc}{\overline{\qquad\quad}}\!\times + \times \Big] + \ldots \right] e^{\Delta} \\
&= D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3) + \\
&\quad + \frac{ig}{4} \frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} \left[\int \mathrm{d}z\ \times\!\overset{\bigcirc}{\overline{\qquad\quad}}\!\times \right] e^{\Delta} \\
&\quad - \frac{ig}{4!} \frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} \left[\int \mathrm{d}z\ \times \right] e^{\Delta}\,.
\end{aligned} \tag{7.16}$$

We have

$$\begin{aligned}
&\frac{ig}{4} \frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} \left[\int \mathrm{d}z\ \times\!\overset{\bigcirc}{\overline{\qquad\quad}}\!\times \right] e^{\Delta} = \qquad (7.17) \\
&= -\frac{ig}{2} D_F(0) \int \mathrm{d}z \Big[D_F(z - x_1)D_F(z - x_2)D_F(x_3 - x_4) + D_F(z - x_1)D_F(z - x_3)D_F(x_2 - x_4) \\
&\qquad + D_F(z - x_1)D_F(z - x_4)D_F(x_2 - x_3) + D_F(z - x_2)D_F(z - x_3)D_F(x_1 - x_4) \\
&\qquad + D_F(z - x_2)D_F(z - x_4)D_F(x_1 - x_3) + D_F(z - x_3)D_F(z - x_4)D_F(x_1 - x_2) \Big]\,.
\end{aligned}$$

The symmetry factor of this diagram is 12. The last term gives

$$\begin{aligned}
-\frac{ig}{4!} \frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} \left[\int \mathrm{d}z\ \times \right] e^{\Delta} &= \\
= -\frac{ig}{4!} \frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} &\left[\int \mathrm{d}z\, D_F(z - x)J(x)\right]^4 e^{\Delta} = \\
= -ig \int \mathrm{d}z\, D_F(x_1 - z)D_F(x_2 - z)&D_F(x_3 - z)D_F(x_4 - z)\,.
\end{aligned} \tag{7.18}$$

The first term of the order g^0 does not contribute to the non-trivial scattering. The numerical coefficients are easily derived by combinatorics, and this suggest a rather simple way to write a diagram at a given order of perturbation theory. Let us return to our main example of the $\frac{g}{4!}\phi^4$ theory and consider all diagrams with contribute to the four-point function. We deduce them as follows. At order g^n we have n vertices, see figure 7.2 and corresponding to the four-point function we draw four external lines, see figure 7.3.

Figure 7.2: At order g^n there are n vertices.

Figure 7.3: Four external lines of the four-point function.

The four-point function is constructed from the following prediagram figure 7.4. We can now join up all the lines. There are three topologically distinct types of Feynman graphs drawn in figure 7.4. The multiplicities are calculates as follows

1) To get diagram (a) join x_1 to one of the legs of the vertex. There are four ways to do it. Now joint x_2 up to one of the remaining legs – there are three ways. Altogether, there are $4! = 24$ ways to obtain the diagram (a)., which is precisely the coefficient in equation (7.18).

2) To make diagram (b) join x_1 directly to one of the external points x_2, x_3 or x_4. There are three ways to do it. Chose one leg of the vertex and join it up to one of the remaining external legs. There are 4×2 ways to do it. Join one of te remaining legs of the vertex to the one remaining point. Join the remaining two legs together. The total multiplicity is $3 \times 4 \times 2 \times 3 = 12 \times 6$, as in (7.17).

3) The multiplicity of the diagram (c) is $3 \times 3 = 9$. The diagram (c) does not appear because we assume to work with the properly normalized functional $Z[J]$ which produces no vacuum diagram.

In summary, the Feynman rules for ϕ^4 theory are rather simple: every line corresponds to the propagator $D_F(x-y)$; every vertex contributes $-ig$ together with one integration; every diagram must be multiplied by a symmetry factor $S/24!$. The graph (b) is called disconnected and it does not contribute to the S-matrix (it modifies a propagator of one of the particles). Only connected graphs, like $\times$ contribute to $S-1$, that is to the non-trivial part of the S-matrix.

7.3 Generating functional for connected diagrams

It turns out that there exists the generating functional which produces the connected diagrams only! It is called $W[J]$ and it is related to $Z[J]$ by the following formula

$$Z[J] = e^{iW[J]} \quad \rightarrow \quad W[J] = -i \log Z[J] \,.$$

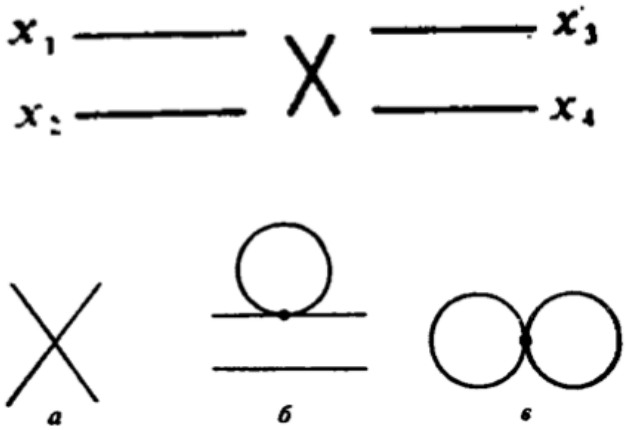

Figure 7.4: Prediagram to construct the four-point function and Feynman diagrams obtained at order g.

The corresponding connected Green's functions are denoted by G_c and they are given by

$$G_c(x_1,\ldots,x_n) = \frac{1}{i^{n-1}}\frac{\delta^n W[J]}{\delta J(x_1)\ldots\delta J(x_n)}\,. \tag{7.19}$$

By using as an example two- and four-point functions let us show that $W[J]$ produces *no* disconnected graphs. We have

$$\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)} = \frac{i}{Z[J]^2}\frac{\delta Z[J]}{\delta J(x_1)}\frac{\delta Z[J]}{\delta J(x_2)} - \frac{i}{Z[J]}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\,. \tag{7.20}$$

Now taking into account that $Z[0]=1$ and $\frac{\delta Z[J]}{\delta J(x)}\Big|_{J=0} = 0$, we find

$$\frac{1}{i}\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)} = -\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)} = \frac{1}{i^2}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)} = G(x_1,x_2)\,. \tag{7.21}$$

This shows that $W[J]$ generates the propagator to any order in g. This is expectable since the propagator has no disconnected parts. To find the four-point function, we differentiate $W[J]$ twice more and put $J=0$ at the end.

First, for the third derivative we find

$$\begin{aligned}\frac{\delta^3 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} &= -\frac{2}{Z[J]^3}\frac{\delta Z}{\delta J(x_1)}\frac{\delta Z}{\delta J(x_2)}\frac{\delta Z}{\delta J(x_3)}\\ &+\frac{i}{Z[J]^2}\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_3)}\frac{\delta Z}{\delta J(x_2)} + \frac{i}{Z[J]^2}\frac{\delta^2 Z}{\delta J(x_2)\delta J(x_3)}\frac{\delta Z}{\delta J(x_1)} + \frac{i}{Z[J]^2}\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)}\frac{\delta Z}{\delta J(x_3)}\\ &-\frac{i}{Z[J]}\frac{\delta^3 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)}\end{aligned}$$

and now applying one more derivative and putting $J=0$, we get up to the order g

$$\begin{aligned}
\frac{\delta^4 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\Big|_{J=0} &= i\Big[-\frac{\delta^4 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}+ \\
&+\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_3)}\frac{\delta^2 Z}{\delta J(x_2)\delta J(x_4)}+\frac{\delta^2 Z}{\delta J(x_2)\delta J(x_3)}\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_4)}+\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)}\frac{\delta^2 Z}{\delta J(x_3)\delta J(x_4)}\Big]= \\
&= i\Big[G(x_1,x_2)G(x_3,x_4)+G(x_1,x_3)G(x_2,x_4)+G(x_1,x_4)G(x_2,x_3)-G(x_1,x_2,x_3,x_4)\Big]= \\
&= \Big[D_F(x_1-x_2)-\frac{ig}{2}D_F(0)\int \mathrm{d}z\, D_F(z-x_1)D_F(z-x_2)\Big]\Big[D_F(x_3-x_4)-\frac{ig}{2}D_F(0)\int \mathrm{d}z\, D_F(z-x_3)D_F(z-x_4)\Big]+ \\
&+\Big[D_F(x_1-x_3)-\frac{ig}{2}D_F(0)\int \mathrm{d}z\, D_F(z-x_1)D_F(z-x_3)\Big]\Big[D_F(x_2-x_4)-\frac{ig}{2}D_F(0)\int \mathrm{d}z\, D_F(z-x_2)D_F(z-x_4)\Big]+ \\
&+\Big[D_F(x_1-x_4)-\frac{ig}{2}D_F(0)\int \mathrm{d}z\, D_F(z-x_1)D_F(z-x_4)\Big]\Big[D_F(x_2-x_3)-\frac{ig}{2}D_F(0)\int \mathrm{d}z\, D_F(z-x_2)D_F(z-x_3)\Big]+ \\
&-[D_F(x_1-x_2)D_F(x_3-x_4)+D_F(x_1-x_3)D_F(x_2-x_4)+D_F(x_1-x_4)D_F(x_2-x_3) \\
&\quad -\frac{ig}{2}D_F(0)\int \mathrm{d}z\Big[D_F(z-x_1)D_F(z-x_2)D_F(x_3-x_4)+D_F(z-x_1)D_F(z-x_3)D_F(x_2-x_4) \\
&\qquad +D_F(z-x_1)D_F(z-x_4)D_F(x_2-x_3)+D_F(z-x_2)D_F(z-x_3)D_F(x_1-x_4) \\
&\qquad +D_F(z-x_2)D_F(z-x_4)D_F(x_1-x_3)+D_F(z-x_3)D_F(z-x_4)D_F(x_1-x_2)\Big] \\
&\qquad -\frac{ig}{4!}\Big[\int \mathrm{d}z\, D_F(x_1-z)D_F(x_2-z)D_F(x_3-z)D_F(x_4-z)+24\text{ terms}\Big]\Big]\,. \qquad (7.22)
\end{aligned}$$

The result is

$$\frac{\delta^4 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\Big|_{J=0} = -\frac{ig}{4!}\Big[\int \mathrm{d}z\, D_F(x_1-z)D_F(x_2-z)D_F(x_3-z)D_F(x_4-z)+24\text{ terms}\Big]\,,$$

which is the sum of connected graphs only.

7.4 Connected diagrams in the momentum space

It is very convenient to work with Green's functions in the momentum space as it allows one to straightforwardly define the vertex functions – one of the most important notions in quantum field theory. Green's functions in the momentum space are defined as the Fourier transform of the coordinate space Green's functions

$$G(p_1,\ldots,p_n)=\int\prod_{i=1}^{n}\mathrm{d}x_i\, e^{ip_1x_1+\ldots p_nx_n}\,G(x_1,\ldots,x_n)\,. \qquad (7.23)$$

Since we are interested in the connected Green's functions below we compute the two- and four-point connected functions in the momentum space up to the order g. We have

$$\begin{aligned}
G_c(p_1,p_2) &= \int \mathrm{d}x_1\mathrm{d}x_2\, e^{ip_1x_1+ip_2x_2}\underbrace{\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{ie^{ik(x_1-x_2)}}{k^2-m^2+i\epsilon}}_{D_F(x_1-x_2)}+ \\
&+ \frac{g}{2}D_F(0)\int \mathrm{d}x_1\mathrm{d}x_2\, e^{ip_1x_1+ip_2x_2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i\,e^{-ik(x_1-x_2)}}{(k^2-m^2+i\epsilon)^2}+\ldots
\end{aligned}$$

Taking into account that the δ-function in d-dimensions has the following Fourier transform

$$\delta(x)=\int\frac{e^{ikx}}{(2\pi)^4}\mathrm{d}x\,,$$

we find

$$G_c(p_1,p_2) = (2\pi)^4\delta(p_1+p_2)\left[\frac{i}{p_1^2-m^2+i\epsilon} + \frac{i}{p_1^2-m^2+i\epsilon}\frac{\frac{g}{2}D_F(0)}{i}\frac{i}{p_1^2-m^2+i\epsilon} + \ldots\right].$$

Analogously, for the four-point function at the leading order g we compute

$$G_c(p_1,p_2,p_3,p_4) \quad = \quad -ig\int\Big(\prod_{i=1}^{4}\int \mathrm{d}x_i\, e^{ip_ix_i}\Big)\int \mathrm{d}z\,\prod_{i=1}^{4} D_F(x_i-z)\,.$$

Performing the Fourier transform, we find

$$G_c(p_1,p_2,p_3,p_4) = (2\pi)^4\delta(p_1+p_2+p_3+p_4)\left[-ig\prod_{i=1}^{4}\frac{i}{p_i^2-m^2-i\epsilon} + \ldots\right]. \tag{7.24}$$

7.5 Self-energy and vertex function

We saw in the previous sections how to apply perturbation theory to calculating the 2- and 4-point Green's functions. We also found that the mass of a particle, defined as a pole of the two-point function, was no longer the bare mass m^2 but $m^2+\delta m^2$, with $\delta m^2 = -\frac{g}{2}D_F(0)$ in ϕ^4 theory. Here we want to look at the problem of summing to all orders and, therefore, obtaining the *exact* Green's functions. We will approach this problem again in ϕ^4 theory.

As was already mentioned, to study scattering processes, we are most interested in *connected* (also called irreducible) Feynman graphs, which are generated by the functional $W[J]$. All the graphs generated by $Z[J]$ can be expressed via connected ones entirely through the formula $Z[J] = e^{iW[J]}$. More precisely, if we denote by $G^{(n)}$ the n-point Green's function and by $G_c^{(n)}$ the corresponding connected one, then $G^{(n)}$ is equal to $G_c^{(n)}$ plus products of $G^{(m)}$, connected Green's point functions of lower order $m<n$.

We therefore made a digression from the class of all Feynman diagrams to the class of connected ones. Now we make a further digression from the class of connected diagrams to the class of *one-particle irreducible* ones. Ignoring numerical prefactors, the connected 2-point function is, to all orders,

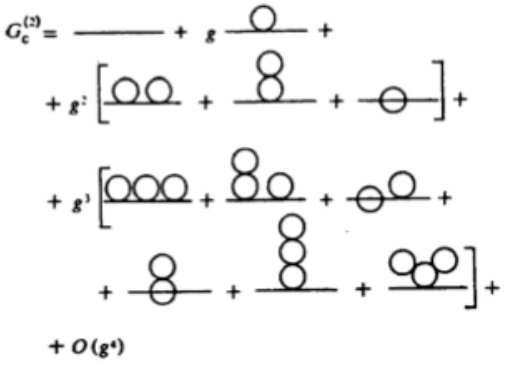

Figure 7.5: Connected 2-point Green's function at all orders in g.

We want to develop a method for summing up all these graphs. The sum will be naturally called the *complete or dressed* propagator. The effect of all interacting graphs is to change the physical mass away from the bare mass and, therefore, give rise to self-energy. Note first that all graphs contain two free external propagators. This allows us to define truncated graphs by multiplying the external legs with inverse propagators. For instance at order g^2 we will have three truncated graphs Of the three truncated graphs of order g^2, the first is a product of of graphs of the lower order,

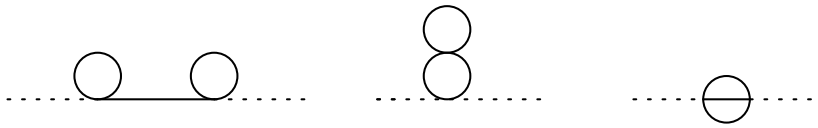

Figure 7.6: Truncated 2-point graphs at order g^2.

but the other two are not; this is because the first graphs contains the propagator. It is called one-particle reducible, while the other two graphs – one-particle irreducible. In general, a one-particle irreducible graph (1PI) is the graph which cannot be make disconnected by removing one line (*i.e.* one propagator). Based on this classification, we may define the *proper self-energy* part as a sum of all 1PI graphs, see figure 7.7. The complete propagator in the momentum space may therefore be

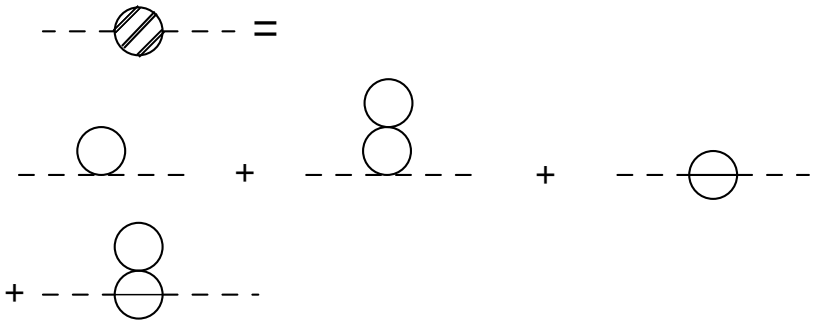

Figure 7.7: One-particle irreducible graphs contributing to the self-energy $\frac{1}{i}\Sigma$.

written in terms of the bare propagator $G_0 = \frac{i}{p^2-m^2}$ and the proper self-energy $\frac{1}{i}\Sigma(p)$ as follows

$$\begin{aligned}
\overline{G}_c^{(2)}(p) &= G_0(p) + G_0(p)\frac{\Sigma(p)}{i}G_0(p) + G_0(p)\frac{\Sigma(p)}{i}G_0(p)\frac{\Sigma(p)}{i}G_0(p) + \ldots = \\
&= G_0(p)\left[1 + \frac{\Sigma(p)}{i}G_0(p) + \frac{\Sigma(p)}{i}G_0(p)\frac{\Sigma(p)}{i}G_0(p) + \ldots\right] = \\
&= G_0(p)\left[1 - \frac{\Sigma(p)}{i}G_0(p)\right]^{-1} = \left[G_0^{-1}(p) - \frac{\Sigma(p)}{i}\right]^{-1} = \\
&= \frac{i}{p^2 - m^2 - \Sigma(p)}\,. \qquad (7.25)
\end{aligned}$$

Here the function $\overline{G}^{(2)}(p)$ is defined as

$$G_c(p_1, p_2) = (2\pi)^4\delta(p_1 + p_2)\overline{G}^{(2)}(p)\,, \quad p = p_1\,,$$

i.e. by stripping off from the momentum space expression $G_c(p_1, p_2)$ the delta function $\delta(p_1 + p_2)$.

Defining the physical mass $m_{\rm phys}$ by the pole of the complete propagator

$$\overline{G}_c^{(2)}(p) = \frac{i}{p^2 - m_{\rm phys}^2}$$

gives

$$m_{\rm phys}^2 = m^2 + \Sigma(p)\,,$$

which justifies the term "self energy" for $\Sigma(p)$. It represents a change from the bare mass to the physical one to all orders in perturbation theory. It happens not because we have divergent diagrams rather because the theory is interacting.

We see that

$$\Big[\overline{G}_c^{(2)}(p)\Big]^{-1} = G_0^{-1}(p) - \frac{\Sigma(p)}{i} = \frac{1}{i}\Big[p^2 - m^2 - \Sigma(p)\Big]\,.$$

This leads to

$$\overline{G}_c^{(2)}(p)\overline{\Gamma}^{(2)}(p) \equiv \overline{G}_c^{(2)}(p)\Big[p^2 - m^2 - \Sigma(p)\Big] = i\,. \tag{7.26}$$

The inverse of the complete propagator is called the *vertex function* $\overline{\Gamma}^{(2)}(p)$ and it contains the inverse of the free propagator plus the sum of 1PI graphs:

$$\overline{\Gamma}^{(2)}(p) = p^2 - m^2 - \Sigma(p)\,. \tag{7.27}$$

Let us show that there exists a generating functional for the functions $\Gamma^{(n)}$. It is denoted by $\Gamma[\phi]$ and is defined by means of the Legendre transform

$$W[J] = \Gamma[\phi] + \int {\rm d}x\, J(x)\phi(x)\,. \tag{7.28}$$

One gets

$$\frac{\delta W[J]}{\delta J(x)} = \phi(x)\,, \qquad \frac{\Gamma[\phi]}{\delta\phi(x)} = -J(x)\,. \tag{7.29}$$

For the propagator we

$$G_c(x_1, x_2) = \frac{1}{i}\frac{\delta^2 W}{\delta J(x_1)\delta J(x_2)} = -i\frac{\delta\phi(x_1)}{\delta J(x_2)}\,. \tag{7.30}$$

Define the kernel

$$\Gamma(x_1, x_2) = \frac{\delta^2\Gamma[\phi]}{\delta\phi(x_1)\delta\phi(x_2)} = -\frac{\delta J(x_1)}{\delta\phi(x_2)}\,. \tag{7.31}$$

It is easy to see that this kernel is inverse to the propagator

$$\int {\rm d}z\, G_c(x_1, z)\Gamma(z, x_2) = \int {\rm d}z\, \frac{\delta\phi(x_1)}{\delta J(z)}\frac{\delta J(x_1)}{\delta\phi(x_2)} = i\frac{\delta\phi(x_1)}{\delta\phi(x_2)} = i\delta(x_1, x_2)\,. \tag{7.32}$$

We define the inverse Fourier transform of $G_c(x_1, x_2)$

$$\begin{aligned} G_c(x_1, x_2) &= \int \frac{{\rm d}p_1 {\rm d}p_2}{(2\pi)^8} e^{-i(x_1 p_1 + x_2 p_2)}(2\pi)^4\delta(p_1 + p_2)\overline{G}_c^{(2)}(p_1) = \\ &= \int \frac{{\rm d}p}{(2\pi)^4} e^{-ip(x_1 - x_2)}\overline{G}_c^{(2)}(p) \end{aligned}$$

and also of the vertex function

$$\begin{aligned}\Gamma(x_1,x_2) &= \int \frac{\mathrm{d}p_1\mathrm{d}p_2}{(2\pi)^8} e^{-i(x_1p_1+x_2p_2)}(2\pi)^4\delta(p_1+p_2)\overline{\Gamma}^{(2)}(p_1) = \\ &= \int \frac{\mathrm{d}p}{(2\pi)^4} e^{-ip(x_1-x_2)}\overline{\Gamma}^{(2)}(p)\,.\end{aligned}$$

Plugging these expressions in the left hand side of (7.32) and integrating over z, we find

$$\int \mathrm{d}z\, G_c(x_1,z)\Gamma(z,x_2) = \int \frac{\mathrm{d}p}{(2\pi)^4} e^{-ip(x-y)}\overline{G}_c^{(2)}(p)\Gamma^{(2)}(p) = i\int \frac{\mathrm{d}p}{(2\pi)^4} e^{-ip(x-y)}\,, \tag{7.33}$$

where we have written the delta-function on the right hand side of (7.32) via its Fourier image. The relation we just obtained implies the relation (7.26).

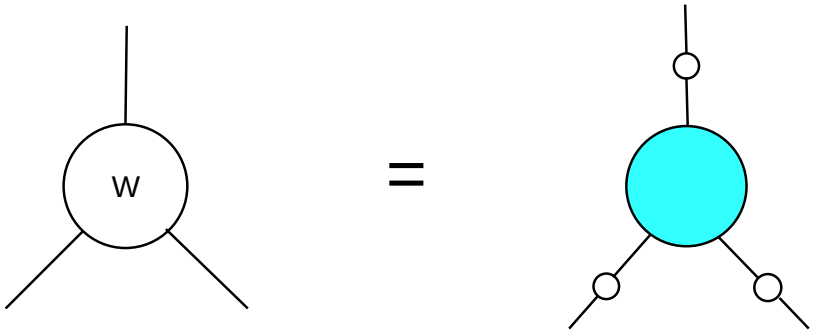

Figure 7.8: The relation between the connected three-point function $G_c^{(3)}$ and the vertex function $\Gamma^{(3)}$. The connected Green's function is nothing else as the vertex function with external lines being the dressed propagators.

Consider the equality (7.32) again and write it in the form

$$\int \mathrm{d}z\, \frac{\delta^2 W[J]}{\delta J(x_2)\delta J(z)}\frac{\delta^2\Gamma[\phi]}{\delta\phi(z)\delta\phi(z')} = -\delta(x_2-z') \tag{7.34}$$

We have the following identity

$$\frac{\delta}{\delta J(x)} = \int \mathrm{d}z\, \frac{\delta\phi(z)}{\delta J(x)}\frac{\delta}{\delta\phi(z)} = i\int \mathrm{d}z\, G_c(z,x)\frac{\delta}{\delta\phi(z)}\,.$$

We take a variation of equation (7.35) over $J(x'')$ and use the identity above

$$\begin{aligned}&\int \mathrm{d}z\, \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(z)}\frac{\delta^2\Gamma[\phi]}{\delta\phi(z)\delta\phi(z')} + \int \mathrm{d}z\, \frac{\delta^2 W[J]}{\delta J(x_2)\delta J(z)}\frac{\delta}{\delta J(x_1)}\Big[\frac{\delta^2\Gamma[\phi]}{\delta\phi(z)\delta\phi(z')}\Big] = \\ &= \int \mathrm{d}z\, \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(z)}\Gamma(z,z') + \int \mathrm{d}z\mathrm{d}z''\, G_c(x_2,z)C_c(z'',x_1)\frac{\delta^3\Gamma[\phi]}{\delta\phi(z)\delta\phi(z'')\phi(z')} = 0\,.\end{aligned}$$

Multiplying both sides of the last relation with $G_c(z',x_3)$ and integrating over z', we get the following remarkable relation

$$\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} = -\int \mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\, G_c(x_1,z_1)G_c(x_2,z_2)G_c(x_3,z_3)\frac{\delta^3\Gamma[\phi]}{\delta\phi(z_1)\delta\phi(z_2)\delta\phi(z_3)}\,. \tag{7.35}$$

The previous relation means

$$G_c(x_1,x_2,x_3) = \frac{1}{i}\int \mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\, G_c(x_1,z_1)G_c(x_2,z_2)G_c(x_3,z_3)\,\Gamma^{(3)}(z_1,z_2,z_3)\,. \tag{7.36}$$

and its meaning is explained in figure 7.8. The last formula can be inverted with the help of relation (7.32) giving

$$\Gamma^{(3)}(x_1,x_2,x_3) = i\int \mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\,\Gamma(z_1,x_1)\Gamma(z_2,x_2)\Gamma(z_3,x_3)\,G_c^{(3)}(z_1,z_2,z_3)\,. \tag{7.37}$$

Differentiating these relations again we find the expression for the connected four-point function, see figure 8.4.

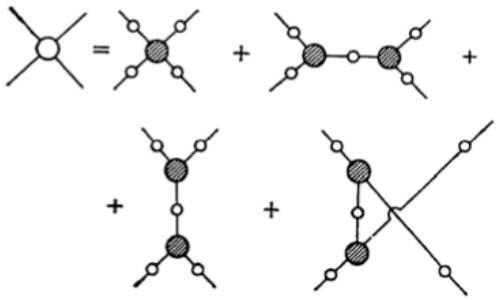

Figure 7.9: The relation between the connected four-point function $G_c^{(4)}$ and the vertex functions $\Gamma^{(3)}$ and $\Gamma^{(4)}$. These vertex functions are 1PI irreducible blocks which are glued by means of dressed propagators.

In general we have

$$\Gamma[\phi] = \sum_{n=0}^{\infty}\frac{1}{n!}\int \mathrm{d}x_1\ldots\mathrm{d}x_n\,\Gamma^{(n)}(x_1\ldots x_n)\,\phi(x_1)\ldots\phi(x_n)\,. \tag{7.38}$$

Since an arbitrary connected diagram is obtained once and only once as a tree diagram using these complete propagators and proper vertices, the proper vertices must be one-particle irreducible amputated n-point functions. Since a tree diagram is never divergent if the vertices and propagators are finite, it is clear that any diagram will be finite if all $\Gamma^{(n)}$ are. Hence the issue of renormalization can be entirely discussed at the level of $\Gamma^{(n)}$.

7.6 Functional methods for fermions

The fermions satisfy the anti-commutation relations

$$\{\psi(x),\psi(y)\} = 0\,.$$

Thus, in quantum theory fermions must be realized as anti-commuting operators. In the approach based on the functional integration, the generating functional for Green's functions is written as a functional integral over the fields, which are regarded as *classical fields*. These classical fields, however, are ant-commuting numbers at each space-time point. As such they satisfy the so-called Grassmann algebra which is generated by letters C_i satisfying the relations

$$\{C_i,C_j\} = C_iC_j + C_jC_i = 0\,,\quad i=1,\ldots,n\,.$$

These relations imply that $C_i^2 = 0$. Any function of C_i can be expanded as

$$f(C_i) = a_0 + a_i C_i + a_{ij} C_i C_j + a_{ijk} C_i C_j C_k + \ldots + a_{1\ldots n} C_1 \ldots C_n \,. \tag{7.39}$$

The operations of multiplication and differentiation satisfy the following rules

$$\left\{ C_i, \frac{\partial}{\partial C_j} \right\} = \delta_{ij} \,, \qquad \left\{ \frac{\partial}{\partial C_i}, \frac{\partial}{\partial C_j} \right\} = 0 \,.$$

We need to define integration with respect to the grassmann variables. The rules are

$$\int \mathrm{d}C_i = 0 \,, \qquad \int \mathrm{d}C_i \, C_i = 1 \,, \tag{7.40}$$

where no summation over i is assumed.

Let us now η and $\bar{\eta}$ be independent complex grassmann variables, so that

$$\int \mathrm{d}\eta = \int \mathrm{d}\bar{\eta} = 0 \,, \qquad \int \mathrm{d}\eta \, \eta = \mathrm{d}\bar{\eta} \, \bar{\eta} = 1 \,.$$

Since, $\eta^2 = \bar{\eta}^2 = 0$, we have $e^{-\eta\bar{\eta}} = 1 - \eta\bar{\eta}$ and, therefore,

$$\int \mathrm{d}\eta \mathrm{d}\bar{\eta} \, e^{-\eta\bar{\eta}} = \int \mathrm{d}\eta \mathrm{d}\bar{\eta} - \int \mathrm{d}\eta \mathrm{d}\bar{\eta} \, \eta\bar{\eta} = 0 + 1 = 1 \,. \tag{7.41}$$

We would like to generalize this formula to the higher dimensional case. For the two-dimensional case we have

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \,, \quad \bar{\eta} = \begin{pmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \end{pmatrix} \,. \tag{7.42}$$

Now we have $\bar{\eta}\eta = \bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2$ and

$$(\bar{\eta}\eta)^2 = (\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2)(\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2) = \bar{\eta}_1\eta_1\bar{\eta}_2\eta_2 + \bar{\eta}_2\eta_2\bar{\eta}_1\eta_1 = 2\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2 \,. \tag{7.43}$$

Thus,

$$e^{-\eta\bar{\eta}} = 1 - (\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2) + \frac{1}{2!} \times 2\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2 \,.$$

Defining he integration measure as

$$\mathrm{d}\eta \, \mathrm{d}\bar{\eta} \equiv \mathrm{d}\bar{\eta}_1 \mathrm{d}\eta_1 \mathrm{d}\bar{\eta}_2 \mathrm{d}\eta_2$$

we have

$$\int \mathrm{d}\eta \mathrm{d}\bar{\eta} \, e^{-\eta\bar{\eta}} = \int \mathrm{d}\bar{\eta}_1 \mathrm{d}\bar{\eta}_2 \, \mathrm{d}\eta_1 \mathrm{d}\eta_2 \, \bar{\eta}_1\eta_1\bar{\eta}_2\eta_2 = 1 \,.$$

Thus the result we get for the two-dimensional case, according to our rules, is the same as in the one-dimensional case! More generally, we could consider the change of variables

$$\eta = M\alpha \,, \quad \bar{\eta} = N\bar{\alpha} \,,$$

where M and N are two 2×2 matrices and α and $\bar{\alpha}$ are the new independent grassmann variables. In particular, we have,

$$\eta_1\eta_2 = (M_{11}\alpha_1 + M_{12}\alpha_2)(M_{21}\alpha_1 + M_{22}\alpha_2) = (M_{11}M_{22} - M_{12}M_{21})\alpha_1\alpha_2 = \det M \alpha_1\alpha_2 \,.$$

However, if we would like to preserve the integration rule

$$\int \mathrm{d}\eta_1 \mathrm{d}\eta_2 \eta_1 \eta_2 = \int \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \, \alpha_1 \alpha_2$$

we must require

$$\mathrm{d}\eta_1 \mathrm{d}\eta_2 = (\mathrm{det}M)^{-1} \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \,,$$

which is very opposite to the logic of the change of variables! We further have

$$(\mathrm{det}MN)^{-1} \int \mathrm{d}\bar{\alpha} \mathrm{d}\alpha \, e^{-\bar{\alpha} N^t M \alpha} = 1 \,.$$

Since $\mathrm{det}MN = \mathrm{det}M^t N$. denoting $M^t N = A$, we have

$$\int \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \, e^{-\bar{\alpha} A \alpha} = \mathrm{det}A \,. \tag{7.44}$$

This is an extremely important formula in the calculus of grassmann variables!

To describe fields with the Fremi statistics, we have to introduce an infinite-dimensional grassmann algebra with generators $C(x)$. The generators obey the anti-commutation relation

$$\{C(x), C(y)\} = 0 \,,$$

and the integration rules

$$\int \mathrm{d}C(x) = 0 \,, \qquad \int C(x) \mathrm{d}C(x) = 1 \,.$$

With these rules we can write the vacuum-to-vacuum transition amplitude for the *free* Dirac field

$$Z_0[\eta, \bar{\eta}] = \frac{1}{N} \int \mathscr{D}\bar{\psi} \mathscr{D}\psi \, \exp\Big[i \int \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x) \Big] \,, \tag{7.45}$$

where the normalization constant is chosen as

$$N = \int \mathscr{D}\bar{\psi} \mathscr{D}\psi \, \exp\Big[i \int \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) \Big] \,.$$

Here $\bar{\eta}(x)$ represents the source term for $\psi(x)$ and $\eta(x)$ the source for $\bar{\psi}(x)$.

Our next goal is to calculate Green's functions in the same manner as has been done for the Klein-Gordon field. Introduce for simplicity the following notation

$$S^{-1} = i\gamma^\mu \partial_\mu - m \,,$$

so that

$$Z_0[\eta, \bar{\eta}] = \frac{1}{N} \int \mathscr{D}\bar{\psi} \mathscr{D}\psi \, \exp\Big[i \int \bar{\psi}(x) S^{-1} \psi(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x) \Big] \,. \tag{7.46}$$

To evaluate this integral, we define

$$Q(\psi, \bar{\psi}) = \bar{\psi} S^{-1} \psi + \bar{\eta}\psi + \bar{\psi}\eta$$

and find the value of ψ which extremizes Q. It is

$$\frac{\delta Q}{\delta \bar{\psi}} = S^{-1}\psi + \eta = 0 \quad \rightarrow \quad \psi_{\mathrm{ext}} = -S\eta \,, \quad \bar{\psi}_{\mathrm{ext}} = -\bar{\eta} S \,.$$

Thus,

$$Q(\psi_{\rm ext},\bar\psi_{\rm ext}) = -\bar\eta S\eta$$

and the path integral takes the form

$$\begin{aligned} Z_0[\eta,\bar\eta] &= \frac{1}{N}\int \mathscr{D}\bar\psi\mathscr{D}\psi\, \exp\Big[i\int \mathrm{d}x\big[-\bar\eta S\eta + (\bar\psi-\bar\psi_{\rm ext})S^{-1}(\psi-\psi_{\rm ext})\big]\Big] \\ &= \frac{1}{N}\exp\Big[-i\int \bar\eta(x)S\eta(y)\mathrm{d}x\mathrm{d}y\Big]\det(-iS^{-1})\,. \end{aligned} \tag{7.47}$$

This gives

$$Z_0[\eta,\bar\eta] = \exp\Big[-i\int \bar\eta(x)S(x-y)\eta(y)\mathrm{d}x\mathrm{d}y\Big]\,,$$

where

$$S(x) = (i\gamma^\mu\partial_\mu + m)D_F(x)$$

is nothing else but the Feynman propagator for the Dirac field. Analogously, to the Klein-Gordon case, we can define the two-point Green's function

$$G(x,y) = \frac{1}{i^2}\frac{\partial^2 Z_0[\eta,\bar\eta]}{\delta\eta(x)\delta\bar\eta(y)}\Big|_{\eta=\bar\eta=0} = iS(x-y)\,. \tag{7.48}$$

The generalization to the interacting case is straightforward

$$Z[\eta,\bar\eta] = \exp\Big[i\int \mathscr{L}_{\rm int}\Big(\frac{1}{i}\frac{\delta}{\delta\eta},\frac{1}{i}\frac{\delta}{\delta\bar\eta}\Big)\mathrm{d}x\Big]Z_0[\eta,\bar\eta]\,.$$

Chapter 8

Renormalization

If the doors of perception were cleansed everything would appear as it is, infinite.

William Blake
The Marriage of Heaven and Hell

Integration over internal loops in Feynman diagrams often leads to divergent expressions. Thus, the perturbation series is miningless unless we find a way to solve this divergence problem. In this chapter we discuss how this can be done for the ϕ^4 theory. The main idea is to build up the perturbation series order by order in the coupling constant and to show that at any given order physical quantities (such as mass, coupling constant, Green's functions) can always be renormalized to finite values. In the theories like ϕ^4, Quantum Electodynamics (QED) and Quantum Chromodynamics (QCD) the renormalization procedure can be carried out to all orders and, therefore, these theories are *renormalizable*.

8.1 Superficial degree of divergence

As we have seen, the quantity $D_F(0)$ is divergent and it modifies the propagator of the free particle giving a contribution to the self-energy. In the momentum space the corresponding expression is

$$\frac{g}{2} D_F(0) = \frac{g}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \,. \tag{8.1}$$

Obviously, in the spherical coordinate system $\mathrm{d}^4 k$ is proportional to k^3, while denominator grows as

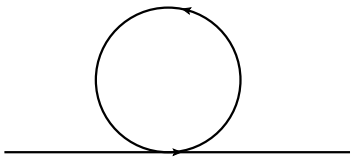

Figure 8.1: Divergent diagram at order g. It has one loop $L = 1$ and one internal line $I = 1$. The superficial degree of divergence is $D = 2$.

k^2, so that the integrand behaves as k for large values of k and, therefore, the integral is quadratically divergent.

This divergence arises for large values of momenta and, for this reason, it is called *ultra-violet*[1]. Another divergent quantity arises at order g^2 it is given by the following expression

$$g^2 \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{\delta(q_1+q_2-p_1-p_2)}{(q_1^2-m^2)(q_2-m^2)} = g^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^8} \frac{1}{(q^2-m^2)((p_1+p_2-q)^2-m^2)} \,. \tag{8.2}$$

This diagram is logarithmically divergent.

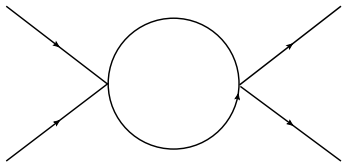

Figure 8.2: Divergent diagram at order g^2, It has one loop $L = 1$ and two internal lines $I = 2$. The superficial degree of divergency $D = 0$.

There is a general way to find a degree of divergence of a particular Feynman graph. Each propagator contributes a power of q^2 in the denominator, each vertex four powers of q in the numerator, together with the momentum conservation δ-function. Also, the number of independent momenta (over which we integrate) is the same as a number of loops. So consider a diagram with of order g^n, *i.e.* a diagram with n vertices, E external lines, I internal lines and L loops and, for generality, consider a theory in d-dimensions. *The superficial degree of divergency* of such a diagram is

$$D = dL - 2I \,.$$

Thus, indeed, with this formula we find for the diagrams discussed $D = 2$ and $D = 0$. We want to express this formula in terms of E and n, *i.e.* we want to eliminate I and L. There are I internal momenta. There is a momentum conservation at each vertex (of which there are n), but there is a total momentum conservation, so there are $n-1$ constraints between the momenta. Hence the number of independent momenta is $I-n+1$ and it concides with the number of loops $L = I-n+1$. In ϕ^4 theory each vertex has four legs; so all together there are $4n$ legs at order n – some of them are external and some of them are internal, internal legs are counted twice as the connect two vertices:

$$4n = E + 2I \,.$$

Thus, we get

$$D = d(I-n+1) - 2I = d(2n - E/2 - n + 1) - (4n - E) = d - \Big(\frac{d}{2} - 1\Big)E + n(d-4) \,.$$

For $d = 4$ we have $D = 4 - E$. This indicates that diagrams with more external legs than 4 will all converge. For instance, if $E = 6$, $D = -2$.

Could we imagine that d is a little bigger than 4? Would it be the case we would run into a terrible situation that the superficial power of divergency increases with the number of loops! This means that adding one more term of perturbation theory leads to more severe divergence. Obviously, to treat such theories from the perturbative standpoint is hopeless. For ϕ^4 in four dimensions we see however that D depends on E only, but not on the order of perturbation theory. Here we are lucky to have only a small number of divergent graphs and the hope is that their effect can be eliminated

[1]Divergences in the Feynman diagrams arising for small values of k are called *infra-red*. We will not touch the issue of the infra-red divergencies here.

by infinite renormalization of various physical quantities. In the case this turns out to be true the corresponding theories are called *renormalizable*. Note that the situation improves if d is slightly less than 4, because in this case the term $n(d-4)$ contributes negatively to D. This fact will be used in the dimensional regularization procedure described in the next section.

If our theory would be ϕ^r, then the corresponding superficial degree of divergency would be

$$D = d - \Big(\frac{d}{2} - 1\Big)E + n\Big[\frac{r}{2}(d-2) - d\Big], \tag{8.3}$$

which for $d = 4$ gives

$$D = 4 - E + n(r-4)\,. \tag{8.4}$$

From this formula we see, for instance, that the ϕ^6 theory is nonrenormalizable, while for ϕ^3 we have $D = 4 - E - n$ and it is called superrenormalizable because D decreases with n meaning that there is only finite number of divergent graphs for a given E. Note also that in two dimensions $D = 2 - 2n$ independent on r.

Let us now return back to $D = 4 - E$ and figure out if all graphs with $E > 4$ are actually convergent.

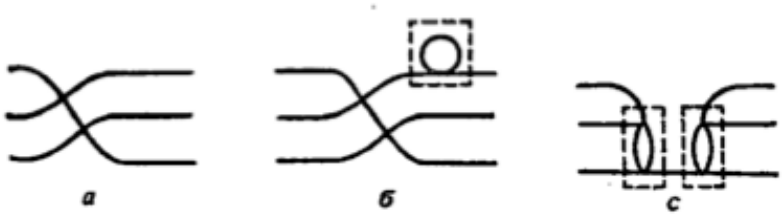

Figure 8.3: The graph a) is convergent. The graphs b) contains the one-loop contribution to the two-point function – it is divergent. Similarly, c) contains two one-loop contributions to the four-point function, it is also divergent.

The situation outlined in figure 8.3 happens to all loop orders. This means that a given Feynman graph diverges if it has hidden two- or four-point functions with one loop (or more) – this is despite of the formula $D = 4 - E$! This is precisely the reason why D is called a a superficial degree of divergency. There is an important *Weinberg's theorem* which says that a Feynman diagram converges if its degree of divergences D, together with the degree of divergence of all its subgraphs, is negative. The two divergent diagrams $G^{(2)}$ and $G^{(4)}$ we discussed above are called primitive. They are the only primitive divergencies in ϕ^4 theory.

8.2 Dimensional regularization

Regularization is a method of isolating the divergencies of the Feynman integrals. Regularization can be performed in a variety of different ways. One extremely intuitive is to introduce a cut-off in the Feynman integrals in the momentum space. For instance, in QED the photon propagator gets modified as

$$\frac{1}{p^2} \to \frac{1}{p^2} - \frac{1}{p^2 - \Lambda^2} = -\frac{\Lambda^2}{p^2(p^2 - \Lambda^2)}$$

As usual, the game relies on the order of limits! Obviously, introduction of Λ improves the convergence properties of the Feynman integrals in the ultra-violet region, but simultaneously such a regularization procedure brings quite a lot of difficulties when non-abelian gauge theories are concerned. One very interesting regularization scheme is that of *dimensional regularization*. The main idea here is to treat the divergent loop integrals as integrals over d-dimensional momenta and take a limit $d \to 4$ at the very end of the calculation only. In this approach the singularities of the Feynman graphs show up as simple poles in the variable $d-4$.

To explain the dimensional regularization scheme, we have first to generalize the Lagrangian description of the system to d dimensions. For the Lagrangian density we have

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi_0\phi^\mu\phi_0 - \frac{m_0^2}{2}\phi_0^2 - \frac{g_0}{4!}\phi^4\,.$$

If we stay with the units $\hbar = 1 = c$ in which the action is dimensionless, then the mass dimension of ϕ is $[\phi] = \frac{d}{2} - 1$. **The coupling constant, also called 'charge', is dimensionless in four dimensions, but if we want to keep it dimensionless in d dimensions, it must be multiplied by μ^{4-d}, where μ is an arbitrary mass parameter.** We will make use of this fact later. We call this $\mathscr{L}$ the bare Lagrangian with the bare field ϕ_0, the bare mass m_0 and the bare charge g_0. As a result of interactions, these bare quantities will undergo a non-trivial renormalization.

One-loop contribution to the self-energy

The one-loop integral contributing to the modification of the propagator is

$$\frac{g_0}{2}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\frac{i}{p^2 - m_0^2 + i\epsilon} \tag{8.5}$$

Due to the fact that $[g_0] = 4 - d$, the mass dimension of this integral is exactly 2.

Thus, we have to learn how to compute this integral in d dimensions. We assume that our space has a Minkowski-like signature with one time- and $(d-1)$ space-like directions. In general, we are interested in the integrals of the type[2]

$$I[d] = \int\frac{\mathrm{d}^d p}{(p^2 + 2pq - m^2)^\alpha}\,, \tag{8.6}$$

where q is another fixed d-dimensional vector. To proceed, we first make a change of $p_\mu \to p_\mu - q_\mu$ so that the integral takes the form

$$I[d] = \int\frac{\mathrm{d}^d p}{[p^2 - (q^2 + m^2)]^\alpha}\,.$$

Second, we to rotate the integration contour through 90^0 (*Wick rotation*) and change the variables $p^0 \to ip^0$. The integral becomes

$$I[d] = i(-1)^\alpha\int\frac{\mathrm{d}^d p}{[p^2 + (q^2 + m^2)]^\alpha} \equiv i(-1)^\alpha\int\frac{\mathrm{d}^d p}{(p^2 + c)^\alpha}\,. \tag{8.7}$$

where we introduce a concise notation $c = q^2 + m^2$. To compute the last integral, we pass to the spherical coordinates in d-dimensions

$$p = (r, \phi, \theta_1, \theta_2, \ldots \theta_{d-2})$$

[2]As our discussion of the integration in d-dimensions is general, we use here the notation m^2 instead of m_0^2.

The measure is

$$\mathrm{d}^d p = r^{d-1}\mathrm{d}r\,\mathrm{d}\phi\prod_{k=1}^{d-2}\sin^k\theta_k\mathrm{d}\theta_k\,.$$

This is the standard formula of the change of variables in the integration measure and below we indicate the integration regions of parameters

$$r<0<\infty\,,\qquad 0<\phi<2\pi\,,\qquad 0<\theta_i<\pi\,.$$

Further, we have (this formula can be, for instance, easily checked with the Mathematica program)

$$\int_0^\pi \sin^k\theta\,\mathrm{d}\theta = \sqrt{\pi}\frac{\Gamma\Big(\frac{k+1}{2}\Big)}{\Gamma\Big(\frac{k+2}{2}\Big)}\,.$$

and, therefore,

$$\prod_{k=1}^{d-2}\int_0^\pi \sin^k\theta_k\,\mathrm{d}\theta_k = \pi^{\frac{d-2}{2}}\frac{\Gamma(1)}{\Gamma(\frac{3}{2})}\frac{\Gamma(\frac{3}{2})}{\Gamma(2)}\frac{\Gamma(2)}{\Gamma(\frac{5}{2})}\cdots\frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})} = \frac{\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d}{2})}\,.$$

Thus,

$$\int\frac{\mathrm{d}^d p}{(p^2+c)^\alpha} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\int_0^\infty\frac{r^{d-1}\mathrm{d}r}{(r^2+c)^\alpha} = \frac{2\pi^{\frac{d}{2}}c^{\frac{d}{2}-\alpha}}{\Gamma(\frac{d}{2})}\int_0^\infty \mathrm{d}t\,t^{d-1}(1+t^2)^{-\alpha}\,, \tag{8.8}$$

where we made a change of variables $t=r/\sqrt{c}$. The formula for the Euler beta-function

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^\infty \mathrm{d}t\,t^{2x-1}(1+t^2)^{-x-y}\,, \tag{8.9}$$

valid for $\mathrm{Re}\,x>0$ and $\mathrm{Re}\,y>0$, so putting

$$x=\frac{d}{2}\,,\qquad y=\alpha-\frac{d}{2}\,,$$

we have

$$\int\frac{\mathrm{d}^d p}{(p^2+c)^\alpha} = \frac{\pi^{\frac{d}{2}}c^{\frac{d}{2}-\alpha}\Gamma\Big(\alpha-\frac{d}{2}\Big)}{\Gamma(\alpha)}\,. \tag{8.10}$$

This is one of the main formulas of dimensional regularization. Thus, we finally obtain

$$I[d] = (-1)^\alpha i\pi^{\frac{d}{2}}\frac{\Gamma\Big(\alpha-\frac{d}{2}\Big)}{\Gamma(\alpha)}\frac{1}{(q^2+m^2)^{\alpha-\frac{d}{2}}}\,. \tag{8.11}$$

One comment is in order. For the case $\alpha=1$ we are most interested in, $y=1-\frac{d}{2}=1-2+\frac{\epsilon}{2}=-1+\frac{\epsilon}{2}$, which is less than zero for $\epsilon\to 0^+$. Thus, for ϵ in the vicinity of zero the integral (8.8) remains divergent and dimensional regularization cannot make it finite. This is, of course, is the consequence of the fact that in four dimensions this integral is quadratically divergent. Strictly speaking, only logarithmically divergent integrals can be made finite by dimensional regularization with $d=4-\epsilon$. To understand this issue, we compute the integral (8.8) for $\alpha=1$ by putting also a momentum cutoff Λ. For $\mathrm{Re}\,d>0$ the integral exists and is evaluated in terms of the Gauss hypergeometric function ${}_2F_1$

$$\int\frac{\mathrm{d}^d p}{p^2+c} = \frac{2\pi^{\frac{d}{2}}c^{\frac{d}{2}-1}}{\Gamma(\frac{d}{2})}\int_0^{\frac{\Lambda}{\sqrt{c}}}\mathrm{d}t\,\frac{t^{d-1}}{1+t^2} = \frac{2\pi^{\frac{d}{2}}c^{\frac{d}{2}-1}}{\Gamma(\frac{d}{2})}\times\Big(\frac{\Lambda}{\sqrt{c}}\Big)^d\frac{1}{d}\,{}_2F_1\Big(1,\frac{d}{2};\frac{2+d}{2};-\frac{\Lambda^2}{c}\Big)\,. \tag{8.12}$$

The large Λ expansion gives

$$\int \frac{\mathrm{d}^d p}{p^2+c} = \frac{2\pi^{\frac{d}{2}} c^{\frac{d}{2}-1}}{\Gamma(\frac{d}{2})}\left[\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{d-2k}\Big(\frac{\Lambda}{\sqrt{c}}\Big)^{d-2k} + \frac{1}{2}\Gamma(1-\tfrac{d}{2})\Gamma(\tfrac{d}{2})\right]. \tag{8.13}$$

Thus, the answer contains two terms: the first one depending on Λ and the second, Λ-independent term, which precisely coincides with the expression (8.10) with $\alpha = 1$! In the renormalization procedure one first considers the limit $\Lambda \to \infty$ while keeping ϵ fixed. As is clear from eq.(8.13), the series in Λ has a structure

$$\frac{1}{2-\epsilon}\Big(\frac{\Lambda}{\sqrt{c}}\Big)^{2-\epsilon} + \frac{1}{\epsilon}\Big(\frac{\Lambda}{\sqrt{c}}\Big)^{-\epsilon} + \dots$$

For $\epsilon > 0$ in the limit $\Lambda \to \infty$ only the first term matters. This term is regular in the limit $\epsilon \to 0$. One normally sets up a scheme, where a first step is to renormalize m^2 by removing the Λ^2-divergence while keeping ϵ finite. As a second step, one performs a multiplicative ϵ-renormalization making Green's functions finite in the limit $\epsilon \to 0$. Usually in the literature on dimensional regularization the first step is assumed as already implemented and therefore is not discussed. In our treatment below we undertake the same root.

Returning back to our one-loop integral in d-dimensions, we find

$$\frac{g_0}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{i}{p^2-m_0^2+i\epsilon} = \frac{ig_0}{2(2\pi)^d}\times I[d]_{\alpha=1,q=0} = \frac{gm_0^2}{32\pi^2}\Big(\frac{4\pi^2\mu^2}{m_0^2}\Big)^{2-\frac{d}{2}}\Gamma\Big(1-\frac{d}{2}\Big),$$

where we introduce a dimensionless coupling g through the relation $g_0 = g\mu^{4-d}$, where μ is an arbitrary mass parameter which represents a freedom of dimensional regularization.

The gamma function has poles at negative integers where

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\Big[\frac{1}{\epsilon} + \psi_1(n+1) + \mathcal{O}(\epsilon)\Big], \tag{8.14}$$

with

$$\psi_1(n+1) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \gamma$$

and $\gamma = -\psi_1(1) = 0.577$ being the Euler-Mascheroni constant. Taking $\epsilon = 4-d$, we find

$$\Gamma\Big(1-\frac{d}{2}\Big) = \Gamma\Big(-1+\frac{\epsilon}{2}\Big) = -\frac{2}{\epsilon} - 1 + \gamma + \mathcal{O}(\epsilon). \tag{8.15}$$

Thus,

$$\begin{aligned}\frac{g\mu^{4-d}}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i}{k^2-m^2+i\epsilon} &= \frac{gm^2}{32\pi^2}e^{\frac{\epsilon}{2}\log\left(\frac{4\pi^2\mu^2}{m^2}\right)}\Gamma\Big(1-\frac{\epsilon}{2}\Big) \overset{\epsilon\to 0}{=} \\ &\overset{\epsilon\to 0}{=} -\frac{gm_0^2}{16\pi^2\epsilon} + \underbrace{\frac{gm_0^2}{16\pi^2}\Big(-1+\gamma-\log\Big(\frac{4\pi^2\mu^2}{m_0^2}\Big)\Big)}_{\text{finite}} + \mathcal{O}(\epsilon)\,.\end{aligned} \tag{8.16}$$

We clearly see that in dimensional regularization the divergence of the present integral manifests itself as a pole in ϵ. Further, the finite part depends on an arbitrary mass scale μ; changing $\mu \to t\mu$ we can adjust the finite part to take *any* desired value! From this result we immediately read off the contribution to the proper self-energy

$$\Sigma = -\frac{gm^2}{16\pi^2\epsilon} + \text{finite} \tag{8.17}$$

and therefore the vertex function can be written up to $\mathcal{O}(\epsilon)$ as

$$\Gamma^{(2)}(p) = p^2 - m_0^2\left[1 - \frac{g}{16\pi^2\epsilon} + \frac{g}{16\pi^2}\Big(-1+\gamma-\log\Big(\frac{4\pi^2\mu^2}{m_0^2}\Big)\Big)\right]. \tag{8.18}$$

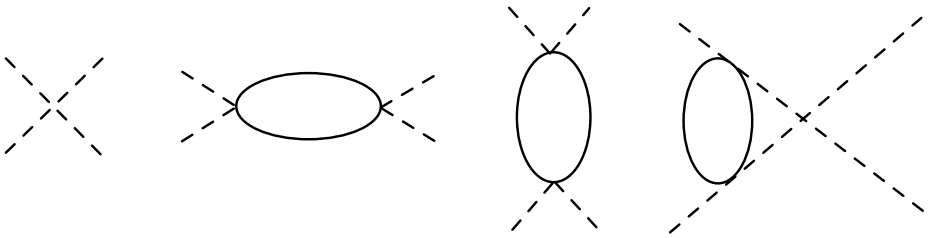

Figure 8.4: The vertex function $\Gamma^{(4)}$ at the leading and sub-leading orders in perturbation theory. Dashed lines depict the amputated propagators. The three graphs at order g^2 are parametrized by the Mandelstam variables s, t, u, respectively. At the leading order the vertex is simply $g\mu^\epsilon$.

One-loop contribution to the four-point vertex function

Now we turn to the one-loop divergent integral at order g_0^2

$$\frac{1}{2}g_0^2\int\frac{\mathrm{d}^dq}{(2\pi)^d}\frac{1}{(q^2-m_0^2+i\epsilon)((p_1+p_2-q)^2-m_0^2+i\epsilon)}\,. \tag{8.19}$$

Since $g_0=g\mu^{4-d}$ the integral has the mass dimension $2(4-d)+d-4=4-d=\epsilon$. Using the formula

$$\frac{1}{ab}=\int_0^1\frac{\mathrm{d}x}{[ax+b(1-x)]^2}\,,$$

we rewrite the integral as

$$\frac{1}{2}g^2(\mu^2)^{4-d}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^dq}{(2\pi)^d}\frac{1}{[q^2+(p^2-2pq)(1-x)-m_0^2+i\epsilon]^2}\,, \tag{8.20}$$

where $p=p_1+p_2$. Making the shift of the integration variable $q\to q+(1-x)p$, the integral is brought to the form

$$\frac{1}{2}g^2(\mu^2)^{4-d}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^dq}{(2\pi)^d}\frac{1}{[q^2+p^2x(1-x)-m_0^2+i\epsilon]^2}\,. \tag{8.21}$$

After the Wick rotation it becomes

$$\frac{i}{2}g^2(\mu^2)^{4-d}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^dq}{(2\pi)^d}\frac{1}{[q^2+m_0^2-p^2x(1-x)]^2}\,. \tag{8.22}$$

The integral over momenta is computed by using the formula (8.10), so we get

$$\begin{aligned}&\frac{i\pi^{\frac{d}{2}}}{2(2\pi)^d}g^2(\mu^2)^{4-d}\Gamma\Big(2-\frac{d}{2}\Big)\int_0^1\mathrm{d}x\,[m_0^2-p^2x(1-x)]^{\frac{d}{2}-2}=\\&\quad=\frac{ig^2(\mu^2)^{2-\frac{d}{2}}}{32\pi^2}\Gamma\Big(2-\frac{d}{2}\Big)\int_0^1\mathrm{d}x\Big[\frac{m_0^2-p^2x(1-x)}{4\pi\mu^2}\Big]^{\frac{d}{2}-2}\,.\end{aligned} \tag{8.23}$$

Here we written the integrand in such a way that it is dimensionless. Detection of the divergent and the finite piece is easy. We have

$$\begin{aligned}&\frac{ig^2\mu^\epsilon}{32\pi^2}\Big(\frac{2}{\epsilon}-\gamma\Big)\int_0^1\mathrm{d}x\,e^{-\frac{\epsilon}{2}\Big[\frac{m_0^2-p^2x(1-x)}{4\pi\mu^2}\Big]}=\\&\quad=\frac{ig^2\mu^\epsilon}{16\pi^2\epsilon}-\frac{ig^2\mu^\epsilon}{32\pi^2}\left(\gamma+\int_0^1\mathrm{d}x\,\ln\Big[\frac{m_0^2-p^2x(1-x)}{4\pi\mu^2}\Big]\right)+\mathcal{O}(\epsilon)\,.\end{aligned}$$

For $0 \leqslant x \leqslant 1$ the maximum of $x(1-x)$ is reached at $x = 1/2$ and it equals $1/4$. Thus, for $p^2 > 4m^2$ the argument of the logarithm is on the branch cut of the latter and the integral representation above looses its validness. Note that the presence of μ^ϵ reflects the correct mass dimension of the original expression[3]. In what follows it is convenient to adopt the notation

$$F(s, m, \mu) = \int_0^1 \mathrm{d}x \, \ln\Big[\frac{m^2 - sx(1-x)}{4\pi\mu^2}\Big], \tag{8.24}$$

where s in the Mandelstam variable $s = (p_1 + p_2)^2$. The other two Mandelstam variables are $t = (p_1 + p_4)^2$ and $u = (p_1 + p_3)^2$. The Mandelstam variables are not independent; if all incoming momenta are on-shell and because of the conservation law they satisfy the relation

$$s + t + u = 4m^2 \,. \tag{8.25}$$

In the region $-\infty < p^2 < 4m^2$ the integral can be computed to give

$$\int_0^1 \mathrm{d}x \, \ln\Big[\frac{m^2 - p^2x(1-x)}{4\pi\mu^2}\Big] = \ln\frac{m^2}{4\pi\mu^2} - 2 + 2\frac{\sqrt{4m^2 - p^2}}{\sqrt{p^2}} \arctan\Big[\frac{\sqrt{p^2}}{\sqrt{4m^2 - p^2}}\Big]. \tag{8.26}$$

This result exhibits a square root branch cut with the branch point at $p^2 = 4m^2$. The expression for the vertex function $\overline{\Gamma}^{(4)}$ reads, *c.f* (7.24),

$$\overline{\Gamma}^{(4)} = -ig\mu^\epsilon + \frac{3ig^2\mu^\epsilon}{16\pi^2\epsilon} - \frac{ig^2\mu^\epsilon}{32\pi^2}\Big(3\gamma + F(s, m_0, \mu) + F(t, m_0, \mu) + F(u, m_0, \mu)\Big), \tag{8.27}$$

where contribution at order g^4 comes from three graphs on figure (8.4).

Renormalization at one loop

At tree level and one loop, up to terms vanishing in the limit $\epsilon \to 0$, we have found for the two- and four-point vertex functions the following result

$$\begin{aligned}
\overline{\Gamma}^{(2)}(p) &= p^2 - m_0^2\left[1 - \frac{g}{16\pi^2\epsilon} + \frac{g}{16\pi^2}\Big(-1 + \gamma - \log\Big(\frac{4\pi^2\mu^2}{m_0^2}\Big)\Big)\right], \\
\overline{\Gamma}^{(4)}(p_i) &= -ig_0 + \frac{3ig_0^2\mu^{-\epsilon}}{16\pi^2\epsilon} - \frac{ig_0^2\mu^{-\epsilon}}{32\pi^2}\left[3\gamma + F(s, m_0, \mu) + F(t, m_0, \mu) + F(u, m_0, \mu)\right].
\end{aligned}$$

The vertex functions above depend on the bare quantities m_0^2 and g_0, and on the additional mass parameter μ of dimensional regularization. They exhibit pole-type singularities in the limit $\epsilon \to 0$.

We suppose that at one loop the bare parameters (g_0, m_0^2) are expressed via renormalized parameters (g, m^2) as

$$m_0^2 = m^2\left(1 + \sum_{k=1}^{\infty} \frac{\mathscr{M}_k\Big(g, \frac{m^2}{\mu^2}\Big)}{\epsilon^k}\right) \equiv m^2 Z_m \tag{8.28}$$

$$g_0 = g\mu^\epsilon\left(1 + \sum_{k=1}^{\infty} \frac{\mathscr{G}_k\Big(g, \frac{m^2}{\mu^2}\Big)}{\epsilon^k}\right) \equiv g\mu^\epsilon Z_g \,. \tag{8.29}$$

The functions Z_m and Z_g are dimensionless. In perturbation theory all Z's are series in renormalized dimensionless charge g starting from unity and having poles in ϵ.

[3]Mass dimensions of the graphs we computed is the same as of the corresponding vertex function $\bar{\Gamma}^{(n)}(p_1, p_2, \ldots, p_{n-1}) = \Gamma^{(n)}(p_1, p_2, \ldots, p_n)\delta(p_1 + \ldots p_n)$. For generic n it is $d + n\Big(1 - \frac{d}{2}\Big) = 4 - n + \Big(\frac{n}{2} - 1\Big)\epsilon$.

Let us show how the procedure of renormalization works. We start from $\overline{\Gamma}^{(2)}(p)$ and substitute their m_0^2. Since $\overline{\Gamma}^{(2)}(p)$ is of the first order in g and has a pole in ϵ, considering renormalization at one loop, it is enough to restrict ourselves to

$$m_0^2 = m^2\Big(1+\frac{\mathscr{M}_1 g}{\epsilon}\Big)\,. \tag{8.30}$$

Concerning g, we do not need to do anything here, since the two-point function already depends on the renormalized coupling g. We then have

$$\overline{\Gamma}^{(2)}(p) \;=\; p^2 - m^2\Big(1+\frac{\mathscr{M}_1 g}{\epsilon}\Big)\left[1-\frac{g}{16\pi^2\epsilon}+\frac{g}{16\pi^2}\left(-1+\gamma-\log\Big(\frac{4\pi^2\mu^2}{m^2\big(1+\frac{\mathscr{M}_1 g}{\epsilon}\big)}\Big)\right)\right].$$

According to the ideology of dimensional regularization, *we have to expand this expression in g keeping ϵ finite* up to the order g, as the computation of the corresponding vertex function has been done up to this order. In particular, in the term containing the logarithm it is enough just to take the leading order m^2, as the sub-leading term will produce the contribution of order g and higher. Thus, making this expansion we get

$$\overline{\Gamma}^{(2)}(p) \;=\; p^2 - m^2\left[1+\frac{\mathscr{M}_1 g}{\epsilon}-\frac{g}{16\pi^2\epsilon}+\frac{g}{16\pi^2}\left(-1+\gamma-\log\Big(\frac{4\pi^2\mu^2}{m^2}\Big)\right)\right].$$

Finally, picking up $\mathscr{M}_1 = \frac{1}{16\pi^2}$ we cancel $1/\epsilon$ terms and get the finite vertex function in the limit $\epsilon \to 0$ at one loop. Such a renormalization scheme where Z-functions (8.28) and (8.29) start from identity and they are designed just to cancel poles in ϵ is called *minimal subtraction scheme*[4]. In what follows we confine ourselves to minimal subtractions. Thus, renormalization of mass at one loop in the minimal subtraction scheme is

$$m_0^2 = m^2\left(1+\frac{\frac{1}{16\pi^2}g}{\epsilon}\right). \tag{8.31}$$

Let us turn our attention to the four-vertex function. Here we should substitute in the four-vertex function

$$g_0 \;=\; g\mu^\epsilon\left(1+\frac{\mathscr{G}_1 g}{\epsilon}\right)$$

and expand the result up to the order g^2 (again keeping ϵ finite), which is the order the vertex function was computed. Concerning m_0, it is obviously enough to substitute just the leading order $m_0 = m$. We find

$$\overline{\Gamma}^{(4)}(p_i) \;=\; -ig\mu^\epsilon\left(1+\frac{\mathscr{G}_1 g}{\epsilon}\right)+\frac{3ig^2\mu^\epsilon}{16\pi^2\epsilon}-\frac{ig^2\mu^\epsilon}{32\pi^2}\left[3\gamma+F(s,m,\mu)+F(t,m,\mu)+F(u,m,\mu)\right].$$

Now by choosing $\mathscr{G}_1 = \frac{3}{16\pi^2}$ we cancel the ϵ-pole and get the finite four-point vertex function in the limit $\epsilon \to 0$. Thus, at one loop in the minimal subtraction scheme the coupling constant is renormalized as

$$g_0 \;=\; g\mu^\epsilon\left(1+\frac{\frac{3}{16\pi^2}g}{\epsilon}\right). \tag{8.32}$$

Starting from two loops, it appears that renormalization of the vertex functions cannot be achieved through renormalization of charge and mass only but one has to also invoke renormalization of the

[4]The minimal subtraction scheme was proposed in 't Hooft, Nucl. Phys. B61, 455 (1973).

normalization of the field strength from its bare value ϕ_0 as $\phi_0 = Z_\phi^{1/2}\phi$, where Z_ϕ has the similar structure to eqs.(8.28) and (8.29)

$$Z_\phi = 1 + \sum_{k=1}^{\infty} \frac{\mathscr{Z}_k\left(g, \frac{m^2}{\mu^2}\right)}{\epsilon^k}\,. \tag{8.33}$$

Here $\mathscr{Z}_1$ starts from the order g^2. Renormalization of an arbitrary vertex function is achieved by multiplying it with a proper power of Z_ϕ:

$$\overline{\Gamma}_r^{(n)}(p_i, g, m, \mu; \epsilon) \quad = \quad Z_\phi^{n/2}\,\overline{\Gamma}^{(n)}(p_i, g_0, m_0; \epsilon)\,.$$

Here on the right hand side we have a bare vertex function $\overline{\Gamma}^{(n)}$ which depends on bare parameters g_0, m_0. On the right hand side we have a renormalized vertex function $\overline{\Gamma}_r^{(n)}$. Both vertex functions, on the left and on the right, also depend on ϵ. Renormalized vertex functions at any order of perturbation theory in g remain finite upon taking the limit $\epsilon \to 0$. All infinities are soaked in a proper dependence of the bare parameters on ϵ.

One of the extremely pleasant features of the minimal subtraction scheme is that Z_m, Z_g and Z_ϕ are actually independent on the parameter $\frac{m}{\mu}$, that is they have the form

$$\begin{aligned} m_0^2 &= m^2\left(1 + \sum_{k=1}^{\infty} \frac{\mathscr{M}_k(g)}{\epsilon^k}\right) \equiv m^2 Z_m\,, \\ g_0 &= g\mu^\epsilon\left(1 + \sum_{k=1}^{\infty} \frac{\mathscr{G}_k(g)}{\epsilon^k}\right) \equiv g\mu^\epsilon Z_g\,, \\ Z_\phi &= 1 + \sum_{k=1}^{\infty} \frac{\mathscr{Z}_k(g)}{\epsilon^k}\,. \end{aligned} \tag{8.34}$$

We will not give here the proof of this statement, but mention that it relies on a statement that divergent terms are μ-independent and the whole μ-dependence occurs in the finite parts of Green's functions only.

Summary and relation to the approach based on counterterms

Renormalization is a procedure of canceling the divergences by adjusting the parameters in the action. For ϕ^4 theory we consider the bare Lagrangian in dimension $d = 4 - \epsilon$

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{m_0^2}{2}\phi_0^2 - \frac{g_0}{4!}\phi_0^4\,.$$

The subscript zero indicates bare quantities. Green's functions and vertex functions obtained from this Lagrangian, for instance,

$$\overline{\Gamma}^{(n)}(p_i, g_0, m_0; \epsilon)$$

are finite for finite ϵ but diverge as $\epsilon \to 0$ at any given loop order.

We further rescale the field by writing $\phi_0 = Z_\phi^{1/2}\phi$, where Z is a multiplicative parameter with zero canonical dimension called the wave function renormalization. In terms of the 'renormalized field' ϕ, the Lagrangian is

$$\mathscr{L} \quad = \quad \frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{Z_\phi m_0^2}{2}\phi^2 - \frac{g_0 Z_\phi^2}{4!}\phi^4\,. \tag{8.35}$$

The Green's functions of the quantum field are now obtained by using this Lagrangian in the functional integral. We let Z, m_0, and g_0 be functions of the dimensional regularization parameter ϵ, and we choose these functions (if possible) so that the Green's and vertex functions of ϕ are finite as $\epsilon \to 0$. If this can be done, then we have succeeded in constructing a continuum field theory, and it is termed 'renormalizable'. We will call m_0 the bare mass, and g_0 the bare coupling, and we will call Z the wave-function, or field-strength, renormalization. In the renormalization procedure the dependence on an arbitrary mass parameter μ enters due to dimensional reasons.

An alternative way of viewing the renormalization is to write the Lagrangian as

$$\begin{aligned} \mathscr{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g\mu^{4-d}}{4!}\phi^4 + \\ &+ \frac{1}{2}\delta Z\,\partial_\mu\phi\partial^\mu\phi - \frac{\delta m^2}{2}\phi^2 - \frac{\delta g\mu^{4-d}}{4!}\phi^4\,. \end{aligned} \tag{8.36}$$

We will call the first three terms the *basic* Lagrangian and the last three the *counterterm* Lagrangian. The renormalized mass m and the renormalized coupling g are finite physical quantities held fixed as $\epsilon \to 0$. The fact that the basic Lagrangian does not lead to finite Green's functions means that it is incomplete. The counterterms are then adjusted to cancel the divergences as $\epsilon \to 0$. This form of the Lagrangian is useful in doing perturbation theory; we treat $\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2$ as the free Lagrangian and the remainder as interaction. The expansion is in powers of the renormalized coupling g. The counterterms are expanded in infinite series, each term cancelling the divergences of one specific graph. The form (8.36) also exhibits the fact that the theory has two independent parameters, m and g. The counterterms are functions of m, g, and of ϵ.

The basic ideas of renormalization procedure are

1) The self-interactions of the field create, among other things, dynamical contributions to the mass of the particle, to the potential between particles, and to the coupling of the field to the single particle state. Thus the measured values of these parameters are renormalized relative to the values appearing in the Lagrangian.

2) These contributions, or renormalizations, are infinite, in many cases. The most important theorem of renormalization theory is that they are the only infinities, in the class of theories called 'renormalizable'.

3) The infinities are cancelled by wave-function, mass, and coupling counterterms, so that the net effect of the interactions is finite.

4) To make quantitative the sizes of the infinities, the theory is regularized, for instance, by the method of dimensional regularization. The infinities appear as divergences when the regularization parameter tends to zero.

8.3 Introduction into renormalization group

As we have seen in the previous section, the renormalized and unrenormalized, *i.e. bare*, vertex functions are related as

$$\overline{\Gamma}_r^{(n)}(p_i, g, m, \mu; \epsilon) = Z_\phi^{n/2}\,\overline{\Gamma}^{(n)}(p_i, g_0, m_0; \epsilon)\,, \tag{8.37}$$

$$\overline{\Gamma}^{(n)}(p_i, g_0, m_0; \epsilon) = Z_\phi^{-n/2}\overline{\Gamma}_r^{(n)}(p_i, g, m, \mu; \epsilon)\,. \tag{8.38}$$

Here we keep ϵ finite and recall that g_0 and m_0 are the bare coupling constant and mass, while m and g are their renormalized counterparts. Parameter Z_ϕ is the wave function renormalization. In

these equations we can either regard the bare parameters as functions of the renormalized ones or take the bare parameters as independent variables; in the latter case the renormalized parameters are functions of the bare ones.

It is important to realize that all the bare quantities, vertex functions in particular, *do not depend* on the mass parameter μ and therefore they are invariant under rescaling

$$\mu \to e^s \mu\,, \qquad s \in \mathbb{R}\,.$$

These rescaling form a group known as the *renormalization* group. Why this is the case is clear from the computation procedure – the bare Green's and vertex functions are obtained from the bare Lagrangian and therefore they *must* depend on bare parameters only; as such they do not involve μ. Indeed, look for instance at the formulae (8.5) and (8.19) for the two- and four-vertex functions at one loop.

Remarkably, equation (8.38) shows that while its left hand side is independent on μ, the right hand side exhibits both an explicit and implicit dependence (through g and m) on μ. Therefore, acting on a bare vertex function with the dimensionless operator $\mu\frac{d}{d\mu}$, we must have

$$0 = \mu\frac{d}{d\mu}\overline{\Gamma}^{(n)}(p_i, g_0, m_0; \epsilon) = \mu\frac{d}{d\mu}\Big[Z^{-n/2}\,\overline{\Gamma}^{(n)}_r(p_i, g, m, \mu; \epsilon)\Big]\,, \tag{8.39}$$

which leads to the following differential equation for the renormalized vertex function

$$\left[\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial g}{\partial\mu}\frac{\partial}{\partial g} + \mu\frac{\partial m}{\partial\mu}\frac{\partial}{\partial m} - \frac{n}{2}\mu\frac{\partial \ln Z_\phi}{\partial\mu}\right]\overline{\Gamma}^{(n)}_r(p_i, g, m, \mu; \epsilon) = 0\,. \tag{8.40}$$

Define the following dimensionless quantities

$$\begin{aligned}
\beta\Big(g, \frac{m}{\mu}, \epsilon\Big) &= \mu\frac{\partial g}{\partial\mu}\,,\\
\gamma_m\Big(g, \frac{m}{\mu}, \epsilon\Big) &= \frac{\mu}{m}\frac{\partial m}{\partial\mu}\,,\\
\gamma_\phi\Big(g, \frac{m}{\mu}, \epsilon\Big) &= \mu\frac{\partial \ln Z_\phi}{\partial\mu}\,.
\end{aligned}$$

The equation (8.40) takes the form

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \gamma_m\frac{\partial}{\partial m} - \frac{n}{2}\gamma_\phi\right]\overline{\Gamma}^{(n)}_r(p_i, g, m, \mu; \epsilon) = 0\,. \tag{8.41}$$

This is a renormalization group equation. It expresses the invariance of $\overline{\Gamma}^{(n)}_r(p_i, g, m, \mu; \epsilon)$ under a change of the mass parameter μ.

It is inconvenient to have in this equation the partial derivative $\mu\frac{\partial}{\partial\mu}$. The latter can be excluded by the following argument. The vertex function $\overline{\Gamma}^{(n)}_r$ must have the same engineering dimension as its bare counterpart, which is

$$D = d + n\Big(1 - \frac{d}{2}\Big) = 4 - n + \epsilon\Big(\frac{n}{2} - 1\Big)\,.$$

It means that under the simultaneous rescaling

$$p_i \to t p_i\,, \quad m \to t m\,, \qquad \mu \to t\mu$$

a vertex function scales as (g is dimensionless!)

$$\overline{\Gamma}^{(n)}_r(t p_i, g, t m, t\mu; \epsilon) = t^D\overline{\Gamma}^{(n)}_r(p_i, g, m, \mu; \epsilon)\,. \tag{8.42}$$

This equation can be rewritten as

$$\overline{\Gamma}_r^{(n)}(tp_i, g, m, \mu; \epsilon) = t^D \overline{\Gamma}_r^{(n)}\left(p_i, g, \frac{m}{t}, \frac{\mu}{t}; \epsilon\right) \tag{8.43}$$

which implies in turn the following differential equation

$$\left[\mu\frac{\partial}{\partial\mu} + m\frac{\partial}{\partial m} + t\frac{\partial}{\partial t} - D\right]\overline{\Gamma}_r^{(n)}(tp_i, g, m, \mu; \epsilon) = 0\,. \tag{8.44}$$

Equation (8.44) can be derived from (8.43) as follows. Applying to (8.43) an operator $t\frac{\partial}{\partial t}$ one gets

$$t\frac{\partial}{\partial t}\overline{\Gamma}_r^{(n)}(tp_i, g, m, \mu; \epsilon) = Dt^D\overline{\Gamma}_r^{(n)}\left(p_i, g, \frac{m}{t}, \frac{\mu}{t}; \epsilon\right) + t^D\, t\frac{\partial}{\partial t}\overline{\Gamma}_r^{(n)}\left(p_i, g, \frac{m}{t}, \frac{\mu}{t}; \epsilon\right). \tag{8.45}$$

Further, one has

$$t\frac{\partial\overline{\Gamma}_r^{(n)}\left(p_i, g, \frac{m}{t}, \frac{\mu}{t}; \epsilon\right)}{\partial t} = \frac{\partial\overline{\Gamma}_r^{(n)}\left(p_i, g, m, \frac{\mu}{t}; \epsilon\right)}{\partial m}\Big|_{m\to\frac{m}{t}}\Big(-\frac{m}{t}\Big) + \frac{\partial\overline{\Gamma}_r^{(n)}\left(p_i, g, \frac{m}{t}, \mu; \epsilon\right)}{\partial\mu}\Big|_{\mu\to\frac{\mu}{t}}\Big(-\frac{\mu}{t}\Big).$$

On the other hand,

$$\begin{aligned} m\frac{\partial\overline{\Gamma}_r^{(n)}\left(p_i, g, \frac{m}{t}, \frac{\mu}{t}; \epsilon\right)}{\partial m} &= m\frac{\partial\overline{\Gamma}_r^{(n)}\left(p_i, g, m, \frac{\mu}{t}; \epsilon\right)}{\partial m}\Big|_{m\to\frac{m}{t}}\Big(\frac{1}{t}\Big), \\ \mu\frac{\partial\overline{\Gamma}_r^{(n)}\left(p_i, g, \frac{m}{t}, \frac{\mu}{t}; \epsilon\right)}{\partial\mu} &= \mu\frac{\partial\overline{\Gamma}_r^{(n)}\left(p_i, g, \frac{m}{t}, \mu; \epsilon\right)}{\partial\mu}\Big|_{\mu\to\frac{\mu}{t}}\Big(\frac{1}{t}\Big) \end{aligned}$$

so that

$$t\frac{\partial\overline{\Gamma}_r^{(n)}\left(p_i, g, \frac{m}{t}, \frac{\mu}{t}; \epsilon\right)}{\partial t} = -\left(m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial\mu}\right)\overline{\Gamma}_r^{(n)}\left(p_i, g, \frac{m}{t}, \frac{\mu}{t}; \epsilon\right). \tag{8.46}$$

Plugging this identity into (8.45) gives precisely eq.(8.44).

Since in eq.(8.41) the differential operator does not depend on p_i, it is valid for tp_i as well, and, therefore, excluding from (8.41) and (8.44) the operator $\mu\frac{\partial}{\partial\mu}$, we find

$$\left[-t\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial g} + (\gamma_m - 1)m\frac{\partial}{\partial m} + D - \frac{n}{2}\gamma_\phi\right]\overline{\Gamma}_r^{(n)}(tp_i, g, m, \mu; \epsilon) = 0\,. \tag{8.47}$$

This equation admits a smooth limit $\epsilon \to 0$ in which D reduces to $D = 4 - n$. Mathematically, this equation expresses directly the effect on $\overline{\Gamma}_r^{(n)}$ of scaling up momenta by a factor of t. Note that if $\beta = \gamma_m = \gamma_\phi = 0$, the effect is simply given by the canonical dimension D, as it would be expected from from a naive scaling argument. Because of interactions one has to apply renormalization which leads to non-trivial $\beta, \gamma_m, \gamma_\phi$ and, therefore, to a departure from the naive scaling behavior of Green's functions. It is worth emphasizing that even if we would start with a massless theory, in which case the corresponding Lagrangian is scale invariant, we would find that Green's functions are not scale invariant because of non-vanishing β and γ_ϕ. In other words, renormalization introduces a *scale* in the form of a mass μ in dimensional regularization; typically a classical scale invariant theory leads to a quantum theory where scale invariance is broken.

Let us now explain how one can solve eq.(8.47) in the minimal subtraction scheme where the quantities β, γ_m and γ_ϕ do not depend on the parameter m/μ, *cf.* the discussion around eq.(8.34). The equation (8.47) then reads

$$\left[-t\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g} + (\gamma_m(g) - 1)\, m\frac{\partial}{\partial m} + D - \frac{n}{2}\gamma_\phi(g)\right]\overline{\Gamma}_r^{(n)}(tp_i, g, m, \mu) = 0\,. \tag{8.48}$$

Let us make the chance of variables $t \to 1/t$ under which $t\frac{\partial}{\partial t} \to -t\frac{\partial}{\partial t}$, and further divide the resulting equation by t. We get

$$\left[\frac{\partial}{\partial t} + \frac{\beta(g)}{t}\frac{\partial}{\partial g} + \frac{(\gamma_m(g) - 1)}{t}\, m\frac{\partial}{\partial m} + \frac{D - \frac{n}{2}\gamma_\phi(g)}{t}\right]\overline{\Gamma}_r^{(n)}(t^{-1}p_i, g, m, \mu) = 0\,. \tag{8.49}$$

This is a partial differential equation of the first order for a function of three variables

$$F(t,g,m) \equiv \overline{\Gamma}^{(n)}_r(t^{-1}p_i, g, m, \mu)\,.$$

Such an equation is solved by the method of characteristics. One assumes a parametric dependence of t, g, m on a parameter s such that

$$\frac{dt}{ds} = 1\,, \qquad \frac{dg}{ds} = \frac{\beta(g)}{t}\,, \qquad \frac{dm}{ds} = \frac{(\gamma_m(g)-1)m}{t}\,. \tag{8.50}$$

The first differential equation implies $t = s$, so that the last two equations take the form

$$\frac{dg}{dt} = \frac{\beta(g(t))}{t}\,, \tag{8.51}$$

$$\frac{dm}{dt} = \frac{(\gamma_m(g(t))-1)m}{t} \tag{8.52}$$

Equation (8.49) takes the form

$$\left[\frac{\partial}{\partial t} + \frac{dg}{dt}\frac{\partial}{\partial g} + \frac{dm}{dt}\frac{\partial}{\partial m}\right] F(t,g,m) = -\frac{D - \frac{n}{2}\gamma_\phi(g)}{t} F(t,g,m) \tag{8.53}$$

and, therefore, reduces to an ordinary differential equation

$$\frac{dF}{dt} = -\frac{D - \frac{n}{2}\gamma_\phi(g)}{t} F\,. \tag{8.54}$$

Performing integration we obtain

$$F(t) = Ce^{-\int_1^t \frac{D-\frac{n}{2}\gamma_\phi(g(\tau))}{\tau}\mathrm{d}\tau} = C\,t^{-D}e^{\frac{n}{2}\int_1^t \frac{\gamma_\phi(g(\tau))}{\tau}\mathrm{d}\tau}\,,$$

where $C = F(1)$ is an integration constant. Thus[5],

$$\overline{\Gamma}^{(n)}_r(t^{-1}p_i, g(t), m(t), \mu) = \overline{\Gamma}^{(n)}_r(p_i, g, m, \mu)\,t^{-D}e^{\frac{n}{2}\int_1^t \frac{\gamma_\phi(g(\tau))}{\tau}\mathrm{d}\tau}$$

or rescaling $p_i \to tp_i$,

$$\overline{\Gamma}^{(n)}_r(tp_i, g, m, \mu) = t^D \exp\left[-\frac{n}{2}\int_1^t \frac{\gamma_\phi(g(\tau))}{\tau}\mathrm{d}\tau\right] \overline{\Gamma}^{(n)}_r(p_i, g(t), m(t), \mu)\,.$$

This is an explicit solution solution of the renormalization group equation in terms of 'running' coupling constant $g(t)$ and 'running mass' $m(t)$. Indeed, under a change of scale of external momenta Green's functions scale in a rather non-trivial way: g and m run and besides their engineering dimension D they develop an anomalous overall scaling represented by the exponential term. The solution is controlled by eqs.(8.51) and (8.52), which we rewrite as

$$t\frac{dg}{dt} = \beta(g(t))\,, \quad g(1) = g\,. \tag{8.55}$$

$$t\frac{d\log m}{dt} = \gamma_m(g(t)) - 1\,, \quad m(1) = m\,. \tag{8.56}$$

The function $\beta(g)$ in the first of these equations is called 'beta-function'. Both $\beta(g)$ and $\gamma_m(g)$ can be computed in perturbation theory, again departing from the fact that bare quantities should not depend on μ. For instance, at one loop we found the following result

$$g_0 = g\mu^\epsilon\left(1 + \frac{\frac{3}{16\pi^2}g}{\epsilon}\right)\,. \tag{8.57}$$

[5]Here the initial conditions are $g(1) = g$ and $m(1) = m$.

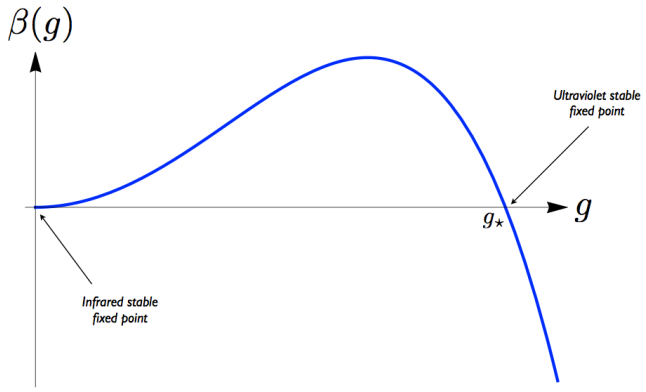

Figure 8.5: Possible form of the β function. Here $g_\star$ is an ultraviolet stable fixed point and $g = 0$ is an infrared stable fixed point. An IR fixed point corresponds to $\beta'(g_\star) > 0$, while a UV one to $\beta'(g_\star) < 0$.

Since g_0 is μ-independent, we must have

$$0 = \mu \frac{dg_0}{d\mu} = \mu \frac{dg}{d\mu} \mu^\epsilon \left(1 + \frac{\frac{3}{16\pi^2} g}{\epsilon}\right) + \epsilon g \mu^\epsilon \left(1 + \frac{\frac{3}{16\pi^2} g}{\epsilon}\right) + \frac{3g}{16\pi^2 \epsilon} \mu^\epsilon \mu \frac{dg}{d\mu}$$

that gives

$$\mu \frac{dg}{d\mu} = \frac{-\epsilon g - \frac{3g^2}{16\pi^2}}{1 + \frac{6g}{16\pi^2 \epsilon}} \,.$$

We now expand the right hand side of this expression in g (keeping ϵ finite!) up to the order g^2

$$\mu \frac{dg}{d\mu} = -\epsilon g + \frac{3g^2}{16\pi^2} + \mathcal{O}(g^3) \tag{8.58}$$

After this is done we can take the limit $\epsilon \to 0$, obtaining

$$\mu \frac{dg}{d\mu} = \frac{3g^2}{16\pi^2} + \mathcal{O}(g^3) \qquad \Longrightarrow \qquad \beta(g) = \frac{3g^2}{16\pi^2} > 0 \,. \tag{8.59}$$

This equation for the running coupling can be rewritten in the form

$$d\left(\frac{1}{g}\right) = -\frac{3}{16\pi^2} d \ln \mu$$

and then integrated to give an expression

$$\frac{1}{g} - \frac{1}{g_s} = -\frac{3}{16\pi^2} \ln \frac{\mu}{\mu_s}$$

from which we finally find the running coupling

$$g(\mu) = \frac{g_s}{1 - \frac{3g_s}{16\pi^2} \ln \frac{\mu}{\mu_s}} \,, \tag{8.60}$$

where $\mu/\mu_s \equiv t$ and an integration constant is chosen such that $g(1) = g_s$.

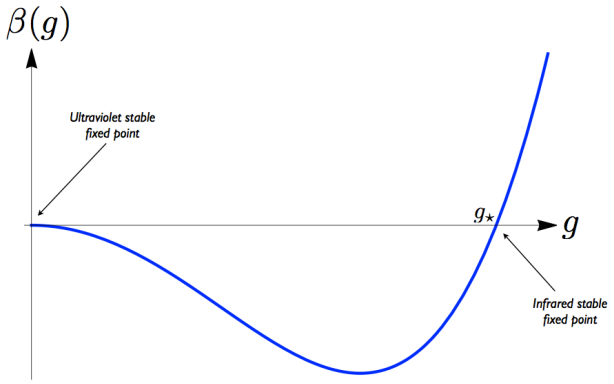

Figure 8.6: Another possible form of the β function. Here $g_\star$ is an infrared stable fixed point, while $g = 0$ is an ultraviolet stable fixed one.

As is clear from (8.60), $g(\mu)$ increases with μ. Indeed, if we start from some small $g_s << 1$ at a given scale μ_s, then the effective coupling will increase with increasing μ. Thus, we will have to deal with larger and larger g so that eventually we will leave the domain of validity of perturbation theory: $g << 1$ or, more exactly, $\frac{3g_s}{16\pi^2} \ln \frac{\mu}{\mu_s} << 1$. At shorter distances, we have to take into account more and more terms on the right hand side of eq.(8.59). This discussion shows that perturbation theory becomes more reliable at large distances (small momenta), that is, in the long range properties of the interaction, and it can be trusted in defining *asymptotic states*. Note that should the sign of the right hand side of (8.59) be negative[6], then perturbation theory would fail for defining the asymptotic states but would work great for short distance behavior. This is precisely the situation which takes place in Quantum Chromodynamics (QCD) – the theory which describes interactions between quarks. Asymptotic states of quarks, like protons, cannot be described by means of perturbation theory. Going to large distances makes interactions between quarks very strong and therefore impossible to describe in the framework of perturbation theory.

β-function scenarios

One can speculate about possible behavior of g outside the domain of perturbation theory. If for some reason even for large μ the running coupling is given by (8.60), then it will blow up at a scale

$$\mu = \mu_s e^{\frac{16\pi^2}{3g_s}} \tag{8.61}$$

which is rather large if g_s is small. This is called *Landau pole* after Landau recognized the same behavior in QED. Of course, there is no reason to believe that the one-loop contribution to $\beta(g)$ is valid for large g.

We do not know how to compute $\beta(g)$ for large g, but we can imagine the following hypothetical behavior of the β-function:

1) For large g the β-function remains positive; with g increasing the beta-function follows a concave or convex curve depending on the sign of $\beta'(g)$. If $\beta(g)$ blows up for some value of g, g itself is infinite there (Landau pole).

2) $\beta(g)$ behaves as in figure 8.5. We assume that it crosses the g-axis at $g_\star$:

$$\beta(g_\star) = 0\,.$$

[6]Notice that this is the case when ϵ is kept finite, see formula (8.58).

The point $g_\star$ is called the fixed point, because if for some reason the coupling was originally at $g_\star$ it would stay there

$$\mu \frac{dg}{d\mu}\Big|_{g_\star} = \beta(g_\star) = 0\,.$$

The behavior of g near $g_\star$ can be analyzed by expanding β around $g_\star$

$$\mu \frac{dg}{d\mu} = (g - g_\star)\beta'(g_\star) + \dots \tag{8.62}$$

We see that the sign of $\beta'(g_\star)$ is crucial. If $\beta'(g_\star) < 0$, then $\frac{dg}{d\mu} > 0$ for g just below g_*, that drives g to a large value, that is, towards the fixed point $g_\star$. For g above $g_\star$, $\frac{dg}{d\mu} < 0$ driving g to smaller values, that is to $g_\star$. This means that g is driven to $g_\star$ as μ increases: such a fixed point is called untraviolet stable, because g will approach the value $g_\star$ asymptotically as $\mu \to \infty$, from above or from below depending on the starting point g_s, which can be either above or below $g_\star$.

3) $\beta(g)$ starts out negative for small g, decreasing its value monotonically. This means that g decreases monotonically with $\ln \mu$. In this case the perturbative approximation becomes better and better at short distances, and g is driven to zero which in this instance is an ultraviolet stable fixed point. Such coupling constant behavior for low g is exhibit by non-abelian gauge theories – a phenomenon known as *asymptotic freedom*.

4) $\beta(g)$ behaves as in figure 8.6. In this case $\beta'(g_\star) > 0$ and $g_\star$ is an infrared point. This means that if at μ_s, $g_s < g_\star$, g will be driven towards zero, but if $g > g_\star$ it will be driven away from $g_\star$ for large values of g.

Chapter 9

Appendices

9.1 Method of stationary phase

Consider an integral of the *Fourier type*

$$F(\lambda) = \int_a^b f(x) e^{i\lambda S(x)} \mathrm{d}x \,.$$

Here $[a, b]$ is a finite interval of $\mathbb{R}$. The function $S(x)$ takes only real values and λ is a large positive number. The function $S(x)$ is called a phase function or simply a phase. We are interested to find an asymptotic behavior of this integral in the limit $\lambda \to +0$.

Typical example of the integral above is the Fourier transform

$$F(\lambda) = \int_a^b f(x) e^{i\lambda x} \mathrm{d}x \tag{9.1}$$

If $f(x)$ is continuous on $[a, b]$, then $F(\lambda) \to 0$ when $\lambda \to +\infty$. Indeed, $\mathrm{Re}(f(x)e^{i\lambda x})$ strongly oscillates for large λ and two neighboring half-waves encompass almost the same but opposite in sign areas. Sum of these areas is small and, as a result, the whole integral is small, *c.f.* figure 9.1. The most general result about asymptotic behavior of such integrals constitute the Riemann-Lebesgue theorem: Let integral $\int_a^b |f(x)| \mathrm{d}x$ converges. Then

$$\int_a^b f(x) e^{i\lambda x} \mathrm{d}x \;\to\; 0 \,, \qquad \lambda \to +\infty \,.$$

The Riemann-Lebesgue theorem contains no information on how fast this integral converges to zero; this depends on differential properties of $f(x)$ and can be actually very slow. Asymptotic expansions of $F(\lambda)$ are possible to obtain only for sufficiently smooth $f(x)$ and $S(x)$. In the following we assume that these two functions are infinite-differentiable.

Let $f(x)$ and $S(x)$ are infinite-differentiable and let $S'(x) \neq 0$ on $[a, b]$. Then $F(\lambda)$ has the following asymptotic expansion as $\lambda \to +\infty$

$$F(\lambda) \sim \frac{e^{i\lambda S(b)}}{i\lambda} \sum_{n=0}^{\infty} \frac{b_n}{(i\lambda)^n} - \frac{e^{i\lambda S(a)}}{i\lambda} \sum_{n=0}^{\infty} \frac{a_n}{(i\lambda)^n} \,, \tag{9.2}$$

where

$$a_n = (-1)^n M^n \Big(\frac{f(x)}{S'(x)} \Big)|_{x=a} \,, \qquad b_n = (-1)^n M^n \Big(\frac{f(x)}{S'(x)} \Big)|_{x=b} \,, \qquad M = \frac{1}{S'(x)} \frac{d}{dx} \,. \tag{9.3}$$

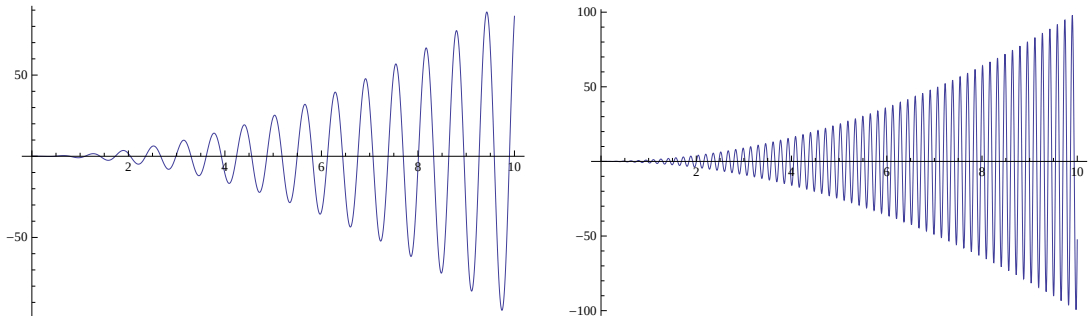

Figure 9.1: An illustation to the Riemann-Lebesgue theorem. On the left figure the function $x^2\cos(10x)$ is plotted in the interval $[0,10]$. On the right figure one has the graph of $x^2\cos(40x)$ on the same interval.

This asymptotic expansion easily follows by integrating by parts $e^{i\lambda S(x)} = \frac{1}{i\lambda S'(x)}\frac{d}{dx}e^{i\lambda S(x)}$ and by using the Riemann-Lebesgue theorem. Note that $F(\lambda)$ expands into an asymptotic series in $1/\lambda$.

As an example consider the integral

$$\Phi(x) = \int_x^\infty e^{it^2}\mathrm{d}t$$

and compute its asymptotics as $x \to +\infty$. By parts,

$$\begin{aligned}\Phi(x) &= \int_x^\infty \frac{1}{2it}d(e^{it^2})\mathrm{d}t = -\frac{e^{-x^2}}{2ix} + \frac{1}{2i}\int_x^\infty e^{it^2}\frac{\mathrm{d}t}{t^2} = \\ &= \frac{ie^{-x^2}}{2x} - \int_x^\infty \frac{1}{4t^3}d(e^{it^2}) = \frac{ie^{-x^2}}{2x} + \frac{e^{ix^2}}{4x^3} - \frac{3}{4}\int_x^\infty e^{it^2}\frac{\mathrm{d}t}{t^4}\,.\end{aligned}$$

Thus,

$$\Phi(x) = e^{ix^2}\Big(\frac{i}{2x} + \frac{1}{4x^3}\Big) + \mathcal{O}\Big(\frac{1}{x^5}\Big)\,.$$

Contribution of a non-degenerate stationary point.

In the previous considerations we assumed that $S'(x) \neq 0$ on $[a,b]$, *i.e.* $S(x)$ has no stationary points on this interval. If there exists stationary points of the phase, then the asymptotic expansion of the integral $F(\lambda)$ changes its form dramatically. For instance $S = x^2$ has a stationary point $x = 0$. Close to this point on the interval of order $1/\sqrt{\lambda}$ the function $\cos(\lambda x^2)$ does not oscillate, while the sum of remaining areas of cosine has the order $\mathcal{O}(1/\lambda)$ which is essentially smaller. In what follows we consider the most interesting case for our applications where $[a,b]$ is replaced by an infinite interval.

To proceed we expand the integrand around a stationary point

$$\begin{aligned}F(\lambda) = \int_{-\infty}^{\infty} f(x)e^{i\lambda x}\mathrm{d}x &= f(x_0)e^{i\lambda S(x_0)}\int_{-\infty}^{+\infty} e^{\frac{i\lambda}{2}S''(x_0)(x-x_0)^2+\cdots}\mathrm{d}x \approx \\ &\approx f(x_0)e^{i\lambda S(x_0)}\int_{-\infty}^{+\infty} e^{\frac{i}{2}\mathrm{sgn}(\mathrm{S''(x_0)})\,\lambda|\mathrm{S''(x_0)}|(\mathrm{x-x_0})^2}\mathrm{d}x\,.\end{aligned}$$

We split the last integral into two and make a choice of variables $t = \sqrt{\lambda|S''(x_0)|}(x - x_0)$. That is

$$F(\lambda) \sim f(x_0)e^{i\lambda S(x_0)}\Big[\int_{-\infty}^{x_0} e^{\frac{i}{2}\mathrm{sgn}(S''(x_0))\,\lambda|S''(x_0)|(x-x_0)^2}\mathrm{d}x + \int_{x_0}^{\infty} e^{\frac{i}{2}\mathrm{sgn}(S''(x_0))\,\lambda|S''(x_0)|(x-x_0)^2}\mathrm{d}x\Big]\,,$$

giving

$$F(\lambda) \sim \frac{f(x_0)e^{i\lambda S(x_0)}}{\sqrt{\lambda|S''(x_0)|}}\Big[\int_{-\infty}^{0} e^{\frac{i}{2}\mathrm{sgn}(S''(x_0))t^2}\mathrm{d}t + \int_{0}^{\infty} e^{\frac{i}{2}\mathrm{sgn}(S''(x_0))t^2}\mathrm{d}t\Big] = \\ = 2\frac{f(x_0)e^{i\lambda S(x_0)}}{\sqrt{\lambda|S''(x_0)|}}\int_{0}^{\infty} e^{\frac{i}{2}\mathrm{sgn}(S''(x_0))t^2}\mathrm{d}t\,.$$

The last integral here is the complete Fresnel integral and it is given by $\sqrt{\frac{\pi}{2}}e^{\frac{i\pi}{4}\mathrm{sgn}(S''(x_0))}$. Thus, the leading term in the asymptotic expansion of $F(\lambda)$ is

$$F(\lambda) = f(x_0)e^{i\lambda S(x_0)}\sqrt{\frac{2\pi}{\lambda|S''(x_0)|}}e^{\frac{i\pi}{4}\mathrm{sgn}(S''(x_0))} + \mathcal{O}\Big(\frac{1}{\lambda}\Big)\,. \tag{9.4}$$

Most importantly, as we see, the presence of the critical point leads to the asymptotic expansion which starts from $1/\sqrt{\lambda}$.

9.2 Path integral for harmonic oscillator

Consider a one-dimensional harmonic oscillator with the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2\,.$$

Equations of motion

$$\ddot{q} + \omega^2 q = 0$$

so that a general solution is

$$q(t) = A\sin\omega t + B\cos\omega t\,.$$

the period of oscillation is $T = \frac{2\pi}{\omega}$. We have then a system of equations

$$\begin{aligned} q_1 &= A\sin\omega t_1 + B\cos\omega t_1\,, \\ q_2 &= A\sin\omega t_2 + B\cos\omega t_2\,. \end{aligned} \tag{9.5}$$

Solving it allows to find A and B and establish a path through q_1 and q_2:

$$q(t) = \frac{1}{\sin\omega(t_2-t_1)}\Big[-q_1\sin(\omega(t-t_2)) + q_2\sin(\omega(t-t_1))\Big]\,.$$

This is a unique well-defined path provided $t_1 - t_2 \neq n\times T/2$, $n\in\mathbb{Z}$. The classical action is

$$S_{\mathrm{cl}} = \frac{m\omega}{2\sin\omega(t_1-t_2)}\Big[(q_1^2+q_2^2)\cos\omega(t_1-t_2) - 2q_1q_2\Big]\,.$$

The path integral reduces

$$W(q_2,t_2;q_1,t_1) = f(t_2-t_1)e^{\frac{i}{\hbar}S_{\mathrm{cl}}}\,,$$

where for f one has a path integral representation

$$f = \int \mathcal{D}q\exp\Big[\frac{im}{2\hbar}\int_{t_1}^{t_2}(\dot{q}^2-\omega^2q^2)\mathrm{d}t\Big]\,,$$

where now $q(t_1) = 0 = q(t_2)$. Expand

$$q(t) = \sqrt{\frac{2}{t_2-t_1}}\sum_{k=1}^{\infty} a_k\sin\pi k\frac{t-t_1}{t_2-t_1}$$

The functions $\sqrt{\frac{2}{t_2-t_1}}\sin\pi k\frac{t-t_1}{t_2-t_1}$ form a complete orthonormal basis on the interval $[t_1,t_2]$ in the space of square-integrable functions with the boundary conditions $q(t_1)=0=q(t_2)$.

We come to the integration over the Fourier coefficients (Jacobian is unessential).

$$f=\lim_{n\to\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\exp\Big[\sum_{k=1}^{n}\frac{im}{2\hbar}\lambda_k a_k^2\Big]\mathrm{d}a_1\dots\mathrm{d}a_n\,.$$

Here

$$\lambda_k=\Big(\frac{k\pi}{t_2-t_1}\Big)^2-\omega^2\,.$$

Now we have to use the Fresnel integral formula

$$\int_{-\infty}^{\infty}e^{i\lambda x^2}\mathrm{d}x=\sqrt{\frac{\pi}{|\lambda|}}e^{\frac{i\pi}{4}\mathrm{sign}\lambda}\,. \tag{9.6}$$

Euler formula

$$\prod_{k=1}^{\infty}\Big|1-\frac{x^2}{k^2\pi^2}\Big|=\frac{|\sin x|}{x}\,,\qquad x>0$$

The correct formula

$$W(q_2,t_2;q_1,t_1)=\Big(\frac{m\omega}{2\pi\hbar|\sin\omega(t_2-t_1)|}\Big)^{1/2}e^{-\frac{i\pi}{4}}e^{\frac{m\omega}{2\sin\omega(t_1-t_2)}\left[(q_1^2+q_2^2)\cos\omega(t_1-t_2)-2q_1q_2\right]}\underbrace{e^{-\frac{i\pi}{2}N}}_{\text{Maslov cor.}}\,. \tag{9.7}$$

valid for

$$n\frac{\pi}{\omega}<t_2-t_1<(n+1)\frac{\pi}{\omega}\,.$$