INSTITUUT VOOR THEORETISCHE FYSICA
UNIVERSITEIT UTRECHT

FINAL EXAM String Theory

Thursday, July 1, 2010, 9:00-12:00, BBL 001.

1) Start every exercise on a separate sheet. Write on each sheet: your name and initials.
2) Please write legibly and clear.
3) The exam consists of **three** exercises, a score is indicated by each question.
4) No lectures notes or any other material (books, calculators, ...) is allowed.
5) This exam counts for 75% of the final mark (the midterm exam for 25%).

1. Closed string spectrum at the selfdual radius. [3points]

Consider the closed bosonic string compactified on a circle of radius R. The space-time is thus $\mathrm{R}^{1,24} \times S^1_R$. We have seen in the lectures that the mass operator for this theory is given by

$$M^2 = \left(\frac{n}{R} - 2mR\right)^2 + 8(N-1) = \left(\frac{n}{R} + 2mR\right)^2 + 8(\tilde{N}-1) \,, \qquad (1)$$

where the integers n and m are the momentum and winding modes. Furthermore, we have defined the number operators in light-cone gauge in terms of the uncompactified transverse string oscillators as (with a sum over i that runs over 24 values)

$$N \equiv \sum_{k>0} (\alpha^i_k)^\dagger \alpha^i_k \; ; \qquad \tilde{N} \equiv \sum_{k>0} (\tilde{\alpha}^i_k)^\dagger \tilde{\alpha}^i_k \,, \qquad (2)$$

which need to satisfy the level-matching conditions $N - \tilde{N} = nm$. In the mass-formula, we have chosen natural units in which the string tension $T = 1/(2\pi\alpha')$ was set equal to $1/(2\pi)$, hence $\alpha' = 1$. The dimension of T is $(length)^{-2}$ and in natural units, the mass M has dimension of inverse length.

- Reintroduce α' into the above formula, in such a way that all dimensions are correct.

- Determine and write down the complete list of massless states in the spectrum, for the special value of the radius

$$R_* = \sqrt{\frac{\alpha'}{2}} \, . \tag{3}$$

- The value of R_* is also called the *selfdual radius*. In terms of T-duality, can you explain what happens at the selfdual radius ? (Be brief, a short answer can suffice).

2. Worldsheet Hamiltonian: $H = L_0$. [4points]

In light-cone coordinates $\sigma^\pm = \tau \pm \sigma$, the action for the superstring in superconformal gauge is

$$S = 2T \int \mathrm{d}^2\sigma \Big[\partial_+ X^\mu \partial_- X_\mu + i(\psi_+^\mu \partial_- \psi_{+,\mu} + \psi_-^\mu \partial_+ \psi_{-,\mu})\Big] \, , \tag{4}$$

where $\psi_\pm$ are the two real components of a Majorana spinor ψ, and $\partial_\pm = \partial/\partial\sigma^\pm$. We can write the action as $S = \int \mathrm{d}^2\sigma \, \mathcal{L}$ with Lagrangian density $\mathcal{L}$. The Hamiltonian density then follows from the general formula $\mathcal{H} = p\dot{q} - \mathcal{L}$, where the momentum p conjugate to q is given by $p \equiv \partial L/\partial \dot{q}$. The Hamiltonian H is then given by the spatial integral of $\mathcal{H}$.

- Show that for the superstring the Hamiltonian density is given by

$$\mathcal{H} = \frac{T}{2}\Big[(\dot{X}^\mu)^2 + (X'^\mu)^2\Big] + iT\Big[\psi_+^\mu \psi'_{+,\mu} - \psi_-^\mu \psi'_{-,\mu}\Big] \, , \tag{5}$$

where the dot $\dot{}$ denotes the time derivative and the prime $'$ denotes the derivative with respect to σ.

For the open superstring, the Hamiltonian is

$$H = \int_0^\pi \mathrm{d}\sigma \, \mathcal{H} \, , \tag{6}$$

and the mode expansions of the fields are

$$\begin{aligned} X^\mu(\tau,\sigma) &= x^\mu + \frac{p^\mu}{\pi T}\tau + \frac{i}{\sqrt{\pi T}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma) \, , \\ \psi_\pm^\mu(\tau,\sigma) &= \frac{1}{2\sqrt{\pi T}} \sum_{r \in Z+\theta} b_r^\mu \mathrm{e}^{-ir\sigma^\pm} \, , \end{aligned} \tag{7}$$

where r runs over the integers in the Ramond sector $\theta = 0$ and over the half integers $\theta = 1/2$ in the Neveu-Schwarz sector.

- In terms of the oscillator modes, show that the Hamiltonian becomes

$$H = \frac{1}{2}\Big(\sum_{n=-\infty}^{\infty} \alpha_n^\mu \alpha_{-n,\mu} - \sum_{r \in Z+\theta} r \, b_r^\mu b_{-r,\mu} \Big) \, , \tag{8}$$

with $\alpha_0^\mu = \frac{1}{\sqrt{\pi T}} p^\mu$. (This expression is in fact the same as one of the Virasoro generators, i.e. $H = L_0$).

3. Super Virasoro algebra. [3points]

Consider the open superstring, with oscillator modes denoted by α_n^μ and b_r^μ. In the quantum theory, the non-vanishing commutation relations are $[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}$, and anti-commutation relations $\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s}$. The super Virasoro constraints are generated by the (normal-ordered) operators

$$
\begin{aligned}
L_m &\equiv \frac{1}{2}\sum_n : \alpha_{-n}^\mu \alpha_{m+n,\mu} : + \frac{1}{2}\sum_r (r + \frac{m}{2}) : b_{-r}^\mu b_{m+r,\mu} : - a\delta_{m,0} \ , \\
G_r &\equiv \sum_n \alpha_{-n}^\mu b_{n+r,\mu} \ ,
\end{aligned}
\tag{9}
$$

where a is the normal ordering constant.

- Prove the commutation relation

$$
[L_m, G_r] = (\frac{m}{2} - r)G_{m+r} \ . \tag{10}
$$