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| EXAMINER: DR. THOMAS W. GRIMM | UTRECHT UNIVERSITY |
| DATE: 13/04/2018 | MIDTERM EXAM |
| TIME: 13:30 - 15:30 | |

# Midterm exam for String Theory

- Write your **name and student number on every sheet**.
- There are four problems. Write your answers to the individual problems on different sheets.
- No lecture notes, books or anything else is allowed. In particular, don't use a pencil for your answers.
- Make sure that your **answers are understandable and readable**. In doubt, explain with a short comment what you're doing.
- On the last page you can find some formulas which might be useful.

## Problem 1: Short questions [10 points]

In this problem we will ask you some basic questions concerning the lecture. You should give **short** answers. Don't lose too much time on Problem 1.

(i) Give reasons why one might want to study string theory? Why is gravity at high energies an exception compared to the other fundamental forces in nature?

(ii) State the Nambu-Goto and the Polyakov action. What is the connection of these two actions?

(iii) Name the symmetries of the Polyakov action.

(iv) What are the consequences of the local symmetries for the energy-momentum tensor $T_{\alpha\beta}$?

(v) Give the Virasoro constraints for the classical and quantized closed bosonic string theory.

(vi) Why does one have to impose the Virasoro constraints? What is their origin?

(vii) Which algebra do the Virasoro generators $L_m$ satisfy? Write down the explicit algebra in terms of commutators.

(viii) Name the level zero and level one states of the closed bosonic string (in the appropriate representations of the little group).

(ix) How many scalars are included on the worldsheet in critical bosonic string theory and critical superstring theory? What are these scalars?

(x) What is the definition of a Weyl transformation? Explain its difference compared to a conformal transformation.

## Problem 2: Classical string [10 points]

Consider the following configuration of a **classical** open string,

$$\begin{aligned} X^0 &= B\tau \\ X^1 &= B\cos(\tau)\cos(\sigma) \\ X^2 &= B\sin(\tau)\cos(\sigma) \\ X^i &= 0\,, \qquad i > 2\,, \end{aligned} \tag{1}$$

where $0 \leq \sigma \leq \pi$ and $B \in \mathbb{R}^+$. In the following you may assume that the wold-sheet metric is fixed to flat gauge, i.e. $h_{\alpha\beta} = \eta_{\alpha\beta}$.

(i) Show that this configuration describes a solution to the equations of motion (following from the Polyakov action) for the field $X^\mu(\tau, \sigma)$ corresponding to an open string with Neumann-Neumann boundary conditions.

(ii) Consider a point on this string at fixed $\sigma$. Calculate the speed of this point via

$$v = \sqrt{\left(\frac{\mathrm{d}X^1}{\mathrm{d}X^0}\right)^2 + \left(\frac{\mathrm{d}X^2}{\mathrm{d}X^0}\right)^2} \tag{2}$$

What is the value of the speed of the endpoints of this string (recall that we work in units $c = 1$)?

(iii) Consider the conserved charges

$$\begin{aligned} P^\mu &= \frac{1}{2\pi\alpha'}\int_0^\pi \mathrm{d}\sigma\, \partial_\tau X^\mu\,, \\ J^{\mu\nu} &= \frac{1}{2\pi\alpha'}\int_0^\pi \mathrm{d}\sigma\, \left(X^\mu\partial_\tau X^\nu - X^\nu\partial_\tau X^\mu\right). \end{aligned} \tag{3}$$

Compute the energy $E = P^0$ and angular momentum $J = J^{12}$ of the string and show that

$$\frac{E^2}{|J|} = \frac{1}{\alpha'}\,. \tag{4}$$

(iv) The energy-momentum tensor of the Polyakov action is in general given by

$$T_{\alpha\beta} = \frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4}h_{\alpha\beta}\, h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu\,. \tag{5}$$

Show explicitly that the solution (1) satisfies the constraint $T_{\alpha\beta} = 0$.

## Problem 3: Open string propagator [8 points]

Recall the mode expansion of the open string with (ND) boundary conditions $X^\mu(\tau, \sigma = \pi) = x_0^\mu$

$$\text{(ND):} \qquad X^\mu(\tau,\sigma) = x_0^\mu + i\sqrt{2\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{1}{r}\alpha_r^\mu\, \mathrm{e}^{-ir\tau}\cos(r\sigma) \tag{6}$$

The propagator is as usual defined by

$$\langle X^\mu(\sigma,\tau)\, X^\nu(\sigma',\tau')\rangle = \mathcal{T}\big[X^\mu(\sigma,\tau)\, X^\nu(\sigma',\tau')\big] - :\, X^\mu(\sigma,\tau)\, X^\nu(\sigma',\tau')\, : \ . \tag{7}$$

The time ordering operator $\mathcal{T}$ is as usual defined by

$$\mathcal{T}\big[A(\tau)\, B(\tau')\big] = \begin{cases} A(\tau)\, B(\tau')\,, & \text{if } \tau > \tau' \\ B(\tau')\, A(\tau)\,, & \text{if } \tau < \tau' \end{cases} \ . \tag{8}$$

(i) Introduce the new coordinates $(z, \bar{z}) \in S^1 \times S^1$ with $z = \mathrm{e}^{i\sigma^-}$ and $\bar{z} = \mathrm{e}^{i\sigma^+}$ $(\sigma^\pm = \tau \pm \sigma)$[1]. Express the mode expansions (6) in terms of the new variables $(z, \bar{z})$.

(ii) Show that the open string propagator with (ND) boundary conditions is given by

$$\begin{aligned}\langle X^\mu(z,\bar{z})\, X^\nu(w,\bar{w})\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu}\Bigg[ &\log\left(\frac{\sqrt{z}-\sqrt{w}}{\sqrt{z}+\sqrt{w}}\right) + \log\left(\frac{\sqrt{\bar{z}}-\sqrt{\bar{w}}}{\sqrt{\bar{z}}+\sqrt{\bar{w}}}\right) \\ &+ \log\left(\frac{\sqrt{z}-\sqrt{\bar{w}}}{\sqrt{z}+\sqrt{\bar{w}}}\right) + \log\left(\frac{\sqrt{\bar{z}}-\sqrt{w}}{\sqrt{\bar{z}}+\sqrt{w}}\right)\Bigg]\end{aligned} \tag{9}$$

**Hint:** You may assume without loss of generality that

$$\mathcal{T}\big[X^\mu(\sigma,\tau)\, X^\nu(\sigma',\tau')\big] = X^\mu(\sigma,\tau)\, X^\nu(\sigma',\tau')\ .$$

You may furthermore assume that $[x_0^\mu,\ \text{anything}] = 0$. You will also need the Taylor series

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{1}{k}(-1)^{k+1}x^k\ . \tag{10}$$

[1] The variables $z$ and $\bar{z}$ are **not** related by complex conjugation.

## Problem 4: Open strings and D-branes [12 points]

In this problem we work with the bosonic string in the critical dimension and in lightcone quantization. We consider a stack of $N$ D3-branes on top of each other and a single D7-brane. These branes occupy the spacetime dimensions according to Table 1. The directions $X^\mu$ filled with a ✓ are occupied by the corresponding branes, whereas a direction marked with a $\times$ is not occupied by the brane. The stack of D3-branes thus extends along $X^\mu$, $\mu = +,-,2,3$ and

| $X^\mu \to$ | + | − | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... | 24 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N$ D3-branes | ✓ | ✓ | ✓ | ✓ | × | × | × | × | × | × | × | × |
| D7-bane | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | × | × | × | × |

Table 1: Brane configuration

the D7-brane occupies the dimensions $X^\mu$, $\mu = +,-,2,\ldots,7$. The D7-brane worldvolume is not transversally separated from the D3-brane worldvolume along their common directions. **You may need some of the information on the next page.**

(i) Consider an open string with both endpoints on the D3-brane stack (D3-D3 string). What is the gauge group in the worldvolume of the D3-brane stack? Compute the mass operator

$$M^2 = 2p^+p^- - \sum_{\text{(NN) directions}} p^a p^a \tag{11}$$

in terms of string oscillators from the condition $L_0 - a_{\text{D3-D3}} = 0$, where $a_{\text{D3-D3}}$ is the normal ordering constant. Compute the normal ordering constant $a_{\text{D3-D3}}$ in this case using $\zeta$-function regularization.

(ii) Construct the massless states with $\alpha' M^2 = 0$ (only the massless ones) and give an interpretation of the state as a field on the D3-brane worldvolume . Give the spectrum of states in the form of a table, as shown below.

| **state** | **worldvolume field** |
|---|---|
| ⋮ | ⋮ |

**Hint:** Don't forget Chan-Paton labels.

(iii) Now consider an open string stretching between the D3-brane stack and the D7-brane (D3-D7 string). The string is attached at $\sigma = 0$ to the D3-branes and at $\sigma = \pi$ to the D7-brane. Compute the mass operator $\alpha' M^2$ and the normal ordering constant $a_{\text{D3-D7}}$ using $\zeta$-function regularization for the D3-D7 string.

(iv) Give the spectrum of states of the D3-D7 string up to and including states with $\alpha' M^2 = \frac{1}{4}$. Also give the interpretation of the states as fields living on the woldvolume of the intersection of the D3-branes and the D7-branes and their multiplicities. You do not have to give representations under the gauge groups this time, i.e. you should answer with a table of the form

| **state** | $\alpha' M^2$ | **worldvolume fields** |
|---|---|---|
| some states | 99 | 38 massive scalars |

# Open string mode expansions:

The mode expansions for the various types of boundary conditions are

$$\text{(NN):}\qquad X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{n}\alpha_n^\mu\, e^{-in\tau}\cos(n\sigma)$$

$$\text{(DD):}\qquad X^\mu(\tau,\sigma) = x_0^\mu + \frac{1}{\pi}(x_1^\mu - x_0^\mu)\sigma + \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{n}\alpha_n^\mu\, e^{-in\tau}\sin(n\sigma)$$

$$\text{(ND):}\qquad X^\mu(\tau,\sigma) = x_1^\mu + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{1}{n}\alpha_n^\mu\, e^{-in\tau}\cos(n\sigma)$$

$$\text{(DN):}\qquad X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{1}{n}\alpha_n^\mu\, e^{-in\tau}\sin(n\sigma)\,.$$

We furthermore identify

$$\alpha_0^\mu = \sqrt{2\alpha'}\,p^\mu \qquad \text{(NN)}$$

$$\alpha_0^\mu = \frac{x_1^\mu - x_0^\mu}{\pi\sqrt{2\alpha'}} \qquad \text{(DD)}\,.$$

# Some useful formulas:

Zeta function regularisation:

$$\sum_{n\in\mathbb{N}} n = -\frac{1}{12}$$

Commutation relations:

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}\,, \qquad [p^+, p^-] = 0$$

In lightcone gauge one has:

$$\alpha_n^+ = \begin{cases} 0\,, & n \neq 0 \\ \sqrt{2\alpha'}\,p^+\,, \qquad \text{for} & n = 0 \end{cases}$$

The Minkowskian inner product in lightcone coordinates reads:

$$A\cdot B = -A^+B^- - A^-B^+ + \delta_{ij}A^iB^j$$