ENDTERM COMPLEX FUNCTIONS

JUNE 28, 2018, 13:30-16:30

- Put your name and student number on every sheet you hand in.
- When you use a theorem, show that the conditions are met.
- Include your partial solutions, even if you were unable to complete an exercise.
- The use of books, notes, computers, calculators, mobile phones, etc., is not allowed.

Exercise 1 (*15 pt*):
Prove that the following integral converges and evaluate it.

$$\int_0^\infty \frac{\cos(\frac{\pi}{2}x)}{x^2-1}\,dx.$$

(*Hint: Use a contour consisting of three semicircles and three segments.*)

Exercise 2 (*15 pt*):
Determine the fractional linear transformations F that map $\mathbb{R}$ to $\mathbb{R}$ and the unit circle to the unit circle. (As you know, the domain of F equals either $\mathbb{C}$ or the complement of exactly one point. The precise meaning of the above is that F maps the real points in its domain to $\mathbb{R}$, and the points on the unit circle in its domain to the unit circle.)

Exercise 3 (*15 pt*):
Let $U \subseteq \mathbb{C}$ be a nonempty open and connected set. A function $f\colon U \to \mathbb{C}$ is *distance preserving* when $|f(z)-f(w)| = |z-w|$ for all z and w in U. Determine (with proof) the distance preserving holomorphic functions on U. (State the theorems (or their names) that you use. As mentioned above: when you use a theorem, show that the conditions are met.)

Exercise 4 (*15 pt*):
Let $S \subset \mathbb{C}$ be a closed set that is discrete (i.e., every point of S is isolated). Let $U \subseteq \mathbb{C}$ be the complement of S. Prove that a holomorphic function f from U to the upper half plane H is necessarily constant.

Exercise 5 (*15 pt*):
Let f be an entire function that is not a polynomial. Show that for every $c \in \mathbb{C}$ there exists an unbounded sequence $(z_n)_{n\in\mathbb{N}}$ such that $f(z_n) \to c$ as $n \to \infty$.

Exercise 6 (*15 pt*):
Prove that the following integral converges and evaluate it.

$$\int_0^\infty \frac{(\log x)^2}{x^2+1}\,dx.$$

(*Hint: Use a contour consisting of two semicircles and two segments. Use an appropriate definition of the complex logarithm.*)