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Elementary Number Theory

Wisb321

Final exam, January 12, 2009, 14-17

During the examination use of notes, books etcetera is not allowed. You can bring a simple calculator to do some of the arithmetic if you want. Motivate your answers. Success!

1. (a) (0.5 pt) Let x, y be two positive co-prime integers (i.e. $\gcd(x, y) = 1$). Show that every odd prime divisor p of $x^2 + y^2$ satisfies $p \equiv 1(\text{mod } 4)$.

   (b) (0.5 pt) Let x, y be two positive integers, not necessarily co-prime this time. Show that $x^2 + y^2$ is divisible by at least one prime p which is either 2 or $1(\text{mod } 4)$ (equivalently, $p \not\equiv 3(\text{mod } 4)$).

   (c) (1 pt) Show that any square z^2 divisible by at least one odd prime p with $p \not\equiv 3(\text{mod } 4)$ can be written as the sum of two positive squares.

   (d) (0.5 pt) We are given that any interval $[m + 1, 2m]$ with $m \in \mathbb{Z}_{\geq 3}$ contains a prime p with $p \equiv 3(\text{mod } 4)$.

   Find all positive integers n such that $n!$ can be written as the sum of two squares.

2. (2.5 pt) Let A, B be positive integers. Assuming the *abc*-conjecture, show that the equation $Ax^4 + By^4 = z^3$ in integers x, y, z with $\gcd(x, y) = 1$ has at most finitely many solutions.

3. Let p be a prime such that $p \equiv 1(\text{mod } 3)$ and denote by ω the cube root of unity $e^{2\pi i/3}$.

   (a) (0.5 pt) Show that for any real a, b we have $|a+b\omega|^2 = a^2 - ab + b^2$.

   (b) (0.5 pt) Show that there exists a Dirichlet character $\chi : (\mathbb{Z}/p\mathbb{Z})^* \to \mathbb{C}$ of order 3.

   (c) (0.5 pt) What is the definition of the Jacobi-sum $J(\chi, \chi)$?

   (d) (0.5 pt) Show that $J(\chi, \chi)$ is a number of the form $a + b\omega$ with $a, b \in \mathbb{Z}$ and explain why the absolute value is $\sqrt{p}$.

   (e) (0.5 pt) Show that p can be written in the form $a^2 - ab + b^2$.

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4. Let $\pi(x)$ be the prime counting function. It is given that

$$\frac{1}{2}\frac{x}{\log x} < \pi(x) < 2\frac{x}{\log x}$$

for all $x > 10$. By p_n we denote the n-th prime. In particular, $\pi(p_n) = n$.

(a) (1 pt) Show that $p_n > n(\log n)/2$ for all $n > 10$.

(b) (1 pt) Show that there exists n_0 such that $p_n < 3n \log n$ for all $n > n_0$.

(c) (0.5 pt) Without using the above inequalities for $\pi(x)$, but only elementary arguments, show that

$$\pi(n) \le \frac{n}{3} + 2$$

for all integers $n \geq 2$.