Geometry and Topology – Exam 1

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1.

1. Show that if $X = X_1 \cup X_2$ is a CW complex, X_1, X_2 are subcomplexes and X_1, X_2 and $X_1 \cap X_2$ are contractible then X is contractible.
2. Show that X is contractible if and only if all maps $f : X \longrightarrow Y$ are null homotopic for every topological space Y.

Exercise 2. For g, n positive integers, let $\Sigma_{g,n}$ be surface obtained by removing n points from the connected sum of g tori. Compute $\pi_1(\Sigma_{g,n})$. Compute the Abelianization $\pi_1(\Sigma_{g,n})/[\pi_1(\Sigma_{g,n}), \pi_1(\Sigma_{g,n})]$.

Exercise 3.

1. Show that if a space X is obtained from a path connected space X_0 by attaching n-cells with $n > 2$ then the inclusion $\iota : X_0 \hookrightarrow X$ induces an isomorphism of fundamental groups: $\iota_* : \pi_1(X_0, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$.
2. Using the previous result or otherwise, compute $\pi_1(\mathbb{R}P^n)$ for $n > 1$ and $\pi_1(\mathbb{C}P^n)$ for $n > 0$.

Exercise 4. A *semigroup* is a set X endowed with a map $m : X \times X \longrightarrow X$ and an element $e \in X$ such that $m(e, x) = m(x, e) = x$ for all x in X. A *topological semigroup* is topological space which is a semigroup and for which the multiplication m is continuous. Following the steps below or otherwise prove that if X is a topological semigroup, then $\pi_1(X, e)$ is an Abelian group.

- Define an operation on loops based at e by

  $$\gamma_1 \star \gamma_2(t) := m(\gamma_1(t), \gamma_2(t)), \qquad \text{for all } \gamma_i : (I, \partial I) \longrightarrow (X, e).$$

  Show that if γ_i' is homotopic to γ_i as loops based at e then $\gamma_1 \star \gamma_2$ is homotopic to $\gamma_1' \star \gamma_2'$. Conclude that $\star$ defines an operation on $\pi_1(X, e)$:

  $$\star : \pi_1(X, e) \times \pi_1(X, e) \longrightarrow \pi_1(X, e).$$

- Letting $\cdot$ denote concatenation of paths and e denote the constant loop, use that $\gamma_1 \simeq \gamma_1 \cdot e$ and $\gamma_2 \simeq e \cdot \gamma_2$ to conclude that $\star$ agrees with the usual product on $\pi_1(X, e)$.
- Using that $\gamma_1 \simeq e \cdot \gamma_1$ and $\gamma_2 \simeq \gamma_2 \cdot e$, conclude that $\pi_1(X, e)$ is Abelian.

Exercise 5. Let $p : \tilde{X} \longrightarrow X$ be a path connected and simply connected covering of X and let $A \subset X$ be a path connected and locally path connected subset. Let $\tilde{A} \subset \tilde{X}$ be a path connected component of $p^{-1}(A)$. Show that $p|_{\tilde{A}} : \tilde{A} \longrightarrow A$ is a covering map corresponding to the kernel of the map $\iota_* : \pi_1(A) \longrightarrow \pi_1(X)$.