Utrecht University
Mathematics
Stochastic processes
Winter 2014

Test, May 27, 2014

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**JUSTIFY YOUR ANSWERS**
**Allowed: Calculator, material handed out in class, *handwritten* notes (*your handwriting*)**
**NOT ALLOWED: Books, printed or photocopied material**

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**NOTE:**

- The test consists of five problems
- The score is computed by adding all the credits up to a maximum of 10 (from a total of 11)

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**Exercise 1.** Consider $n$ events $A_1, A_2, \ldots, A_n$ of a probability space such that $\mathbb{P}(A_i) = 1$ for every $i$.

(a) (0.5 pts.) Prove that $\mathbb{P}\Big(\bigcap_{i\geq 1} A_i\Big) = 1$.

(b) (0.5 pts.) Conclude that the events must have a non-empty intersection.

**Problem 2.** Let $Z_1, Z_2, \ldots$ be independent random variables with the same moment-generating function $\phi_Z(t)$. Let $N$ be an non-negative integer-valued random variable independent from the previous ones. Consider the random variable

$$Y \;=\; \sum_{i=1}^{N} Z_i \;.$$

(a) (0.5 pts.) If $N$ is a Poisson random variable with rate $\lambda$, show that the moment-generating function of $Y$ is

$$\phi_Y(t) \;=\; \exp\big[\lambda\big(\phi_Z(t) - 1\big)\big] \;.$$

(b) (0.5 pts.) More generally, if $\phi_N(t)$ is the moment-generating function of $N$, prove that

$$\phi_Y(t) \;=\; \phi_N\big(\log \phi_Z(t)\big) \;.$$

**Problem 3.** Consider a random walk on the half line with a drift towards the origin. This is a stochastic process $(X_n)_{n\geq 0}$ with image in $\mathbb{N}_{\geq 0}$ whose non-zero transition probabilities are

$$\begin{aligned} P_{i\,i+1} &= p \quad , \; i \geq 0 \\ P_{i\,i-1} &= q \quad , \; i \geq 1 \end{aligned}$$

with $p < q$ and $0 < p + q \leq 1$. Furthermore,

$$P_{01} = p \quad , \quad P_{00} = 1 - p \;.$$

(a) (1 pt.) Compute the stationary probability (invariant measure) $\mathbb{P}$.

(b) (1 pt.) Compute the long-term mean position (mean of the stationary probability). [*Hint*: $\sum_{i\geq 0} i\,\alpha^i = \alpha \frac{d}{d\alpha} \sum_{i\geq 0} \alpha^i$ for $\alpha < 1$.]

(Turn over, please)

**Problem 4.** At an airport travellers arrive following a Poisson process $N(t)$ with rate 1000/hour. As they go through customs, some of them are subjected to a superficial inspection, some to a detailed inspection and some are not inspected at all. The airport opens at 6AM.

(a) (1 pt.) If 1500 passengers have arrived before 8 AM, what will be the mean number of passengers arriving before 10 AM?

(b) There are three officers performing superficial inspections and one performing detailed inspections. Inspection times are independent and exponentially distributed with a mean of 2 minutes for a superficial inspection and 5 minutes for a detailed one. A passenger, due for a detailed inspection, finds the corresponding inspector available while the three "superficial" inspectors are each of them busy with other travellers. Find the probability the passenger subjected to the detailed inspection

-i- (1 pt.) be the *first* to leave?

-ii- (1 pt.) be the *last* to leave?

(c) (1 pt.) The selection procedure is such that 10% of the passengers are randomly chosen for a superficial inspection and a further 1% are chosen for a detailed inspection (that is, 11% of travellers are inspected). Given that in the first ten minutes fifteen passengers have been submitted to the superficial inspection, what is the probability that in the same period exactly 4 passengers have gone through the detailed inspection.

**Problem 5.** After being repaired a machine remains in working condition for an exponentially distributed time with rate $\lambda$. When it fails, its failure is of either of two types. If the failure is of type 1, the repair time of the machine is exponential with rate $\mu_1$, while if it is of type 2 the repair time is exponential with rate $\mu_2$. Each failure is of type 1 with probability $p$ and of type 2 with probability $1 - p$, independently of the time it took the machine to fail.

(a) (1 pt.) Write the Kolmogorov backward equations for the repair process (call "0" the state in which the machine is up and running).

(b) In the long run,

-i- (1 pt.) what proportion of the time is the machine down due to type 1 failure?;

-ii- (1 pt.) what proportion of time is the machine up?