Utrecht University
Mathematics
Stochastic processes
Spring 2011

Test, June 29, 2011

JUSTIFY YOUR ANSWERS

Allowed: material handed out in class and *handwritten* notes (*your handwriting*)

Exercise 1. Consider a branching process with offspring number with mean μ. That means, a sequence of random variables $(X_n)_{n\geq 0}$ with

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i \qquad n \geq 1$$

where Z_n are iid random variables (offspring distribution) independent of the (X_n) with mean μ.

(a) (4 pts.) Show that $E(X_n) = \mu^n \, E(X_0)$

(b) (4 pts.) Assuming that $\mu < 1$ determine the mean total number of descendants of a single individual.

Exercise 2. Consider three independent exponential random variables X_1, X_2 and X_3 with respective rates λ_1, λ_2 and λ_3. Compute

(a) (8 pts.) $E(X_2 \mid X_1 < X_2 < X_3)$.

(b) (8 pts.) $E(X_1 X_2 \mid X_1 < X_2 < X_3)$.

Exercise 3. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ and T an exponential random variable of rate μ, independent of the process. Consider $s < t$,

(a) Find:

-i- (5 pts.) $P\big(N(t) - N(s) \geq 2\big)$.

-ii- (5 pts.) $P\big(N(t) = 20, N(s) = 5, N(s/2) = 3\big)$.

-iii- (5 pts.) $E\big[T N(T)\big]$.

(b) Prove:

-i- (5 pts.) $E\big[N(t) \mid N(s)\big] = N(s) + \lambda(t-s)$.

-ii- (5 pts.) $E\big[N(s) \mid N(t)\big] = N(t)\, s/t$.

Exercise 4. At a border post the interarrival times of travellers are iid exponentially distributed with rate 50/hour. Independently of their arrival, 70% of the travellers are from the European Union, while 30% are non-EU. The post has two desks for EU-travellers, each with exponential service time of rate 20/hr and one desk for non-EU travellers also with exponential service time but with half this rate (the process is longer for non-EU people!).

(a) (5 pts.) Given that in the first ten minutes exactly 8 EU-travellers arrived, what is the probability that in the same period exactly 4 non-EU travellers came to the post?

(b) A non-EU traveller arrives to the post and finds her desk available while in each of the EU-desks there is one person being served. What is the probabilty that out the three travellers

-i- (5 pts.) the non-EU traveller be the *first* to leave?

-ii- (5 pts.) the non-EU traveller be the *last* to leave?