Utrecht University
Mathematics
Stochastic processes
Fall 2012

Test, January 15, 2013

JUSTIFY YOUR ANSWERS

Allowed: material handed out in class and *handwritten* notes (*your handwriting*)

NOTE:

- The test consists of four problems plus two bonus problems
- The score is computed by adding all the credits up to a maximum of 10

Problem 1. Requests to a computer system are handled by two servers that provide answers with respective independent exponential rates λ_1 and λ_2. Requests are processed on a first-come first-serve basis as soon as a server becomes free. Request C arrives and finds both servers busy processing requests A and B. Denote W the waiting time of request C until a server becomes free, T_C its processing time once accepted by a server, and $T = W + T_C$ the total time elapsed between the arrival of request C and the completion of its answer.

(a) (1 pt.) Determine the law of W.

(b) (1 pt.) Prove that

$$E(T) = \frac{3}{\lambda_1 + \lambda_2} .$$

Problem 2. Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Let T_n denote the n-th inter-arrival time and S_n the time of the n-th event. Let $t > 0$. Find:

(a) (0.5 pts.) $P\big(N(t) = 10 \mid N(t/2) = 5, N(t/4) = 3\big)$.

(b) (1 pt.) $P\big(N(t/2) = 5 \mid N(t) = 10\big)$.

(c) (0.5 pts.) $E\big[S_5 \mid S_4 = 3\big]$.

(d) (1 pt.) $E\big[T_2 \mid T_1 < T_2 < T_3\big]$.

Problem 3. Consider a pure-birth process, that is a continuous-time Markov chain characterised by birth rates λ_i, $i = 0, 1, 2, \ldots$, and zero death rates ($\mu_i = 0$ for all $i \geq 0$). Note that, in this case, only upward transitions are allowed, that is $P_{ij}(t) = 0$ if $j < i$.

(a) (1 pt.) Write the Kolmogorov backward equations for the transition evolutions $P_{ij}(t)$ for $i \geq j$ (discriminate the cases $j = i$ and $j \geq i + 1$).

(b) (1 pt.) Determine $P_{ii}(t)$ for $i \geq 0$.

(c) (1 pt.) Assuming $\lambda_i = \lambda$ for $i \geq 0$, determine the transition evolutions $P_{i\,i+1}(t)$ for all $i \geq 0$. Verify that they coincide with those of a Poisson process with rate λ.

(Please turn over)

Problem 4. Consider an exponential queuing system with 2 servers available: Arrival and service times are independent exponential random variables. Customers arrive independently at rate λ and wait in line till the first server becomes available. Each of the two servers processes customers at rate μ.

(a) (0.5 pts.) Write the system as a birth-and-death chain, that is, determine the birth rates λ_n and death rates μ_n.

(b) (1 pt.) Determine the limiting probabilities P_i, $i \geq 0$. Under which condition do these probabilities exist?

(c) (0.5 pts.) Show that if $\lambda = \mu$, in the long run there is at least one server idle 2/3 of the time.

Bonus problems

Bonus 1. (1 pt.) Let T_i, $i \geq 1$ be a sequence of independent identically distributed exponential random variables of rate λ, and let $S_n = \sum_{i=1}^{n} T_i$, $n \geq 1$. Prove that for each $n \geq 1$ and each $t > 0$,

$$P(S_n \leq t\,,\, S_{n+1} > t) = \frac{e^{-\lambda t}\,(\lambda t)^n}{n!}\,.$$

Bonus 2. (1 pt.) Let $(\pi_i)_{0 \leq i \leq n}$ be the invariant measure for the discrete-time Markov process on $S = \{0, 1, \ldots, n\}$ defined by a matrix $(P_{ij})_{0 \leq i,j \leq n}$ with $P_{ii} = 0$. Prove that the measure

$$P_i = \frac{\pi_i/\nu_i}{\sum_j \pi_j/\nu_j} \qquad 0 \leq i \leq n$$

is then invariant for the continuous-time Markov chain with state space S, jump rates ν_i and transition probabilities P_{ij}, $0 \leq i, j \leq n$.