Natuur- en Sterrenkunde, Universiteit Utrecht

# NS-374B: Topics in Astrophysics and Cosmology
## mid-term exam (block 4 2016/17)

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1. Which compact stellar objects are known to be remnants from supernovae explosions? Briefly outline how do we observe them.

2. For each of the three compacts objects studied in the lecture indicate the electromagnetic spectral regions on which their radiation, or radiation from the the regions surrounding them, is stronger.

3. Which type of compact object is associated to type Ia supernovae? What is the property of these compact objects that make them good standard candles?

4. The energy loss of a pulsar can be estimated by treating the neutron star as

   (a) an accelerator of charged particles

   (b) a large rotating magnetic dipole

   (c) a fast cooling thermal conductor

   (d) a large rotating top being slowed down by friction due to its magnetic field

   (e) none of the above

5. White dwarfs and degeneracy of quantum gases

   (a) In a realistic electron degenerate gas the equation of state is not independent of the temperature. Show that the degeneracy criterion can be expressed as $T \ll \frac{\hbar^2 n_e^{2/3}}{m_e k_B}$. Would this condition be more difficult to be realised for a degenerate gas of neutrons and protons? Justify your answer.

   (b) Starting from the structure equations for a non-relativistic white dwarf show that $\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{2/3}}\frac{d\rho}{dr}\right) = -\frac{12\pi G}{5K}\rho,$ where $K$ is a constant.

   (c) Argue from dimensional considerations that a solution of the equation in (b) is of the form $\rho(r) \propto \frac{f(r/R)}{R^6}$, where $R$ is the radius of the compact body.
   Hint: The only parameter with the dimensions of a length in a white dwarf is $R$.

   (d) Show that the mass-volume relation of white dwarfs follows from the result in (c).

6. The crab pulsar as a period of $P = 33$ ms. Let $M$ and $R$ denote its mass and radius.

   (a) Show that the period increasing rate, assuming that the source of energy radiated is mechanical rotational energy, is given by $\dot{P} = \frac{LP^3}{4\pi^2 I}$ where $L$ and $I = \frac{2}{5}MR^2$ and the total luminosity and moment of inertia of the pulsar.

   (b) The total luminosity and the spin slowing down rate have been measured to be $L = 2 \times 10^{31}$ J/s and $\dot{P} = 1.52 \times 10^{-13}$ Hz·s. Using typical pulsar parameters for the value of the mass and radius of the crab pulsar do you think that the assumption used in (a) about the source of energy correct? Which conclusions do you take from your result?

7. Identify three main differences between electromagnetic and gravitational waves.

8. Explain how the solution to linear general relativity in the presence of a time dependent gravity source solves the action at distance problem.

9. Consider a ground based gravitational wave interferometer as discussed in the lectures. The beam lines of the interferometer are along the $x$- and $y$-axis, and the $z$-axis is perpendicular to the detector plane.

   (a) A plus polarised gravitational wave with wave vector $\vec{k} = \frac{k}{\sqrt{2}}(\vec{e}_y + \vec{e}_z)$ and amplitude $h$ strikes the interferometer. Determine the strain along each of the two beam lines.

   (b) Determine the energy flux of this gravitational wave across the plane of the interferometer.

10. Consider a binary star system where both stars have the same mass $M$ and orbital radius $R$ in their circular orbits around the centre of mass of the system. The perturbed metric produced by this system along a direction that makes an angle $\theta$ with the $z$-axis perpendicular to the orbital plan is

$$h_{ij} = -\frac{8GMR^2\Omega^2}{c^4 r}\begin{pmatrix} \cos(2\Omega(t-z/c)) & \cos\theta\sin(2\Omega(t-z/c)) & \sin\theta\sin(2\Omega(t-z/c)) \\ \cos\theta\sin(2\Omega(t-z/c)) & -\cos^2\theta\cos(2\Omega(t-z/c)) & \cos\theta\sin\theta\cos(2\Omega(t-z/c)) \\ \sin\theta\sin(2\Omega(t-z/c)) & \cos\theta\sin\theta\cos(2\Omega(t-z/c)) & -\sin^2\theta\cos(2\Omega(t-z/c)) \end{pmatrix}$$

   (a) Determine the amplitude of both the plus and cross polarisation along the direction $\theta$ as well as their relative phase.

   (b) The total luminosity of this binary system can be compute by integrating the energy for all $\theta$ and $\phi$. Write down this integral and determine the expression for the total luminosity.

## Astronomical and Physical Parameters and Constants

| | |
|---|---|
| $M_\odot = 1.989 \times 10^{30}$ kg | mass of the Sun |
| $R_\odot = 6.9599 \times 10^{8}$ m | radius of the Sun |
| $L_\odot = 3.826 \times 10^{26}$ J/s | luminosity of the Sun |
| $T_\odot = 5770$ K | surface temperature of the Sun |
| $M_\oplus = 5.9736 \times 10^{24}$ kg | mass of the Earth |
| $R_\oplus = 6.371 \times 10^{6}$ m | radius of the Earth |
| 23 h 56 m 0.09054 s | Sidereal day |
| 86400 s | Solar day |
| $3.155815 \times 10^{6}$ s | Sidereal year |
| $3.155693 \times 10^{6}$ s | Tropical year |
| 1 ly $= 9.4605 \times 10^{15}$ m | light-year |
| 1 pc $= 3.0857 \times 10^{16}$ m $= 3.2616$ ly | parsec |
| 1 AU $= 1.496 \times 10^{11}$ m | Astronomical Unit |
| $h = 6.63 \times 10^{-34}$ J·s | Planck's constant |
| $e = 1.60 \times 10^{-19}$ C | elementary charge |
| $G = 6.67 \times 10^{-11}$ N·m$^2$/kg$^2$ | gravitational constant |
| $c = 3.00 \times 10^{8}$ m s$^{-1}$ | speed of light |
| $k_B = 1.38 \times 10^{-23}$ J/K $= 8.617$ eV/K | Boltzmann constant |
| $\sigma_{SB} = 5.67 \times 10^{-8}$ W·m$^{-2}$·K$^{-4}$ | Stefan-Boltzmann constant |
| $m_H = 1.673 \times 10^{-27}$ kg | mass of the proton |
| $m_{He} = 6.643 \times 10^{-27}$ kg | mass of a Helium nucleus |
| $m_e = 9.11 \times 10^{-31}$ kg | electron mass |
| $r_N \approx 10^{-15}$ m $= 1$ fm (Fermi) | approximate size of atomic nucleus |
| $\sigma_T = 6.65 \times 10^{-29}$ m$^2$ | Thomson cross-section |
| $\epsilon_0 = 8.85 \times 10^{-12}$ N·m$^2$/C$^2$ | permittivity of free space |

## Planck Mission best fit data for cosmological parameters (March 2013)

| | |
|---|---|
| $H_0 = 67.8 \pm 0.77$ km s$^{-1}$ Mpc$^{-1}$ | Hubble constant |
| $h = 0.678$ | reduced Hubble constant |
| $t_0 = 13.798 \pm 0.037$ billion years | Age of the Universe |
| $\Omega_{B,0} = 0.048999$ | Baryonic matter density parameter |
| $\Omega_{DM,0} = 0.26709$ | Dark matter density parameter |
| $\Omega_{\Lambda,0} = 0.6825$ | Dark energy density parameter |
| $n_s = 0.9624$ | Spectral index |

# Useful Equations

## Physics of stellar remnants

- Core collapse

$$v_s > v_{\text{infall}}^{(inner)} \propto r \text{ and } v_s < v_{\text{infall}}^{(outer)} \propto 1/\sqrt{r}$$

- probability density distribution function $f(p)$

$$\text{example: } P = \frac{1}{3}\int (vp)\, 4\pi f(p) p^2 dp$$

- Fermi gas parameters

$$p_F = \hbar\,(3\pi^2 n_e)^{1/3} = \hbar\,(3\pi^2 Y_e\, \rho/m_H)^{1/3}$$

$$E_F = \frac{p_F^2}{2m} \text{ for a non-relativistic gas and } p_F c \text{ for a ultra relativistic one}$$

- Thermal Pressure in an ideal gas

$$P = \frac{\rho}{\mu\, m_H} k_B T = n k_B T \quad (k_B = R/N = \text{gas const/Avogadro const})$$

- Pressure in a completely degenerate gas

$$P = \frac{8\pi}{3h^3}\int_0^{p_F} v\, p^3 dp = \frac{8\pi}{3h^3}\int_0^{p_F} \frac{p^4 c^2}{\sqrt{p^2c^2 + m_e^2 c^4}} dp$$

$$\left(\text{from } P = 2\int \frac{1}{3} v p \frac{d^3p}{h^3}\right)$$

- Condition for degeneracy

$$\frac{T}{\rho^{2/3}} < \frac{\hbar^2}{3m_e k}\left(\frac{3\pi^2 Y_e}{m_H}\right)^{2/3} \quad \text{where } Y_e \text{ is the electron to baryon ratio}$$

- Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\,\theta^n$$

- Chandrasekhar Mass

$$M_{\text{Ch}} \approx \frac{\sqrt{6}}{32\pi}\left(\frac{hc}{G}\right)^{3/2}\left(\frac{Y_e}{0.5}\right)^2 \frac{2.018}{m_H^2} \approx 1.457\left(\frac{Y_e}{0.5}\right)^2 M_\odot$$

- Conservative lower bound for the stability of a rotating dense object

$$\rho_{lb} > \frac{3\pi}{GP^2}$$

- Pulsar period increase rate

$$2\pi\frac{\dot{P}}{P^2} I = \frac{2\pi}{3}\frac{B_P^2 R^6 \sin^2\theta}{\mu_0^2 c^3}\frac{8\pi^3}{P^3}$$