INSTITUTE FOR THEORETICAL PHYSICS
UTRECHT UNIVERSITY

Midterm EXAM Field Theory in Particle Physics

Wednesday, April 11, 2018, 10.00 - 12.00, Koningsberger, ATLAS

1) Start every exercise on a **separate** sheet.
2) Write on each sheet: your full name and student number.
3) Please write legibly and clear. *Keep your answers brief and to the point!*
4) The exam consists of **two** exercises.

1. Nonabelian gauge theory with adjoint fermions

a) Consider a fermion field ψ with Lagrangian

$$\mathcal{L}_\psi = -\bar{\psi}\not{D}\psi - m\bar{\psi}\psi\,, \qquad D^\mu = \partial^\mu - gA^\mu\,. \tag{1}$$

By requiring that this Lagrangian is invariant under the *non-abelian* infinitesimal transformation $\delta\psi = g\xi\psi$, derive the transformation rule of the gauge field A. Note that the gauge indices of the fields are contracted in the appropriate manner (suppressed in our notation).

b) Assume that the fermion transforms in the adjoint representation of the gauge group $\delta\psi^a = g f_{bc}{}^a \xi^b \psi^c$. Write out the Lagrangian in (1) making all gauge indices explicit.

c) Given the covariant derivative in (1), work out the field-strength tensor using

$$-gG^{\mu\nu} = [D^\mu, D^\nu]\,. \tag{2}$$

You do not need to make the gauge indices explicit. Show that the field-strength $G^{\mu\nu}$ also transforms in the adjoint representation.

d) In the remainder of this exercise you can assume that the structure constants $f_{abc} = f_{ab}{}^c$ defined through

$$[t_a, t_b] = f_{ab}{}^c t_c \tag{3}$$

are totally anti-symmetric. Show that $\varepsilon^{\mu\nu\rho\sigma} G_{\mu\nu\,a} G_{\rho\sigma}{}^a$ is gauge invariant, and that it can be written as,

$$\varepsilon^{\mu\nu\rho\sigma} G_{\mu\nu\,a} G^a_{\rho\sigma} = 4\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\left(A_{\nu\,a}\partial_\rho A^a_\sigma - \tfrac{1}{3} g f_{abc} A_\nu{}^a A_\rho{}^b A_\sigma{}^c\right). \tag{4}$$

Hint: use that the ε-symbol is completely anti-symmetric, and the Jacobi identity

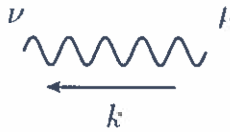
$$f_{abe}f_{cde} - f_{ace}f_{bde} + f_{ade}f_{bce} = 0\,. \tag{5}$$

2. Kaon decay to two photons

Let us consider the Lagrangian for a complex field ϕ, representing π^+ and π^- particles, coupled to photons

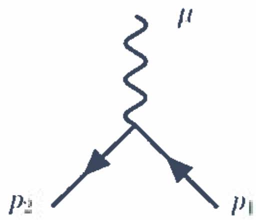
$$\begin{aligned}\mathcal{L}_\phi = & -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \\ & - \mathrm{i}eA_\mu[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi] - e^2A_\mu^2\,\phi^*\phi\,. \end{aligned} \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Feynman rules corresponding to $\mathcal{L}_\phi$ read

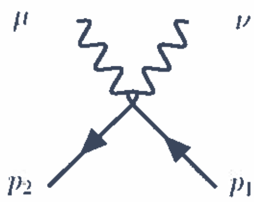


$$= \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2}\left(\eta_{\mu\nu} - \left(1 - \frac{1}{\lambda^2}\right)\frac{k_\mu k_\nu}{k^2}\right)$$

$$= \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{p^2 + m^2}$$



$$= \mathrm{i}(2\pi)^4(-\mathrm{i}e)(\mathrm{i}p_1^\mu + \mathrm{i}p_2^\mu)$$



$$= \mathrm{i}(2\pi)^4(-e^2)(\eta^{\mu\nu})$$

a) Explain, using a covariant derivative, where each of the interaction terms in $\mathcal{L}_\phi$ comes from.

To (1) we add a term for the kaon-pion interaction

$$\mathcal{L}_{K\pi\pi} = gK_S\phi^*\phi\,, \tag{2}$$

where K_S is the kaon field. The kaon K_S is an electrically neutral meson.

b) There are three one-loop diagrams mediated by virtual pions that contribute to $K_S \to \gamma(p) + \gamma(q)$. One of these is drawn in fig. 1, draw the other two diagrams.

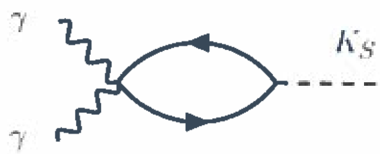


Figure 1: One of the diagrams for $K_S \to \gamma\gamma$.

c) Compute the diagram in fig. 1 and show that the answer in $n = 4 + \varepsilon$ dimensions reads

$$2\eta^{\mu\nu} g e^2 \frac{\pi^{n/2}}{(2\pi)^n} \frac{2}{\varepsilon} \Gamma(1-\varepsilon/2)\, m^\varepsilon \int_0^1 \mathrm{d}x \left[1 - \frac{m_K^2}{m^2} x(1-x)\right]^{\varepsilon/2} . \tag{3}$$

To show this you may use the relations

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} \tag{4}$$

$$\int \frac{\mathrm{d}^n q}{(2\pi)^n} \frac{1}{(q^2+m^2)^\alpha} = \frac{i\pi^{n/2}}{(2\pi)^n} \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)} (m^2)^{n/2-\alpha} , \tag{5}$$

where $\Gamma(z) = \Gamma(1+z)/z$ is the Euler gamma function, and $\Gamma(z) = \frac{1}{z} - \gamma_{\mathrm{E}} + \mathcal{O}(z)$.

d) Show that the result for the invariant amplitude resulting from all three diagrams may be written as

$$\mathcal{M}_{\mu\nu} = \frac{2\,ge^2}{\mathrm{i}(2\pi)^n} \int \mathrm{d}^n k \frac{(2\,k+p)_\mu\,(2\,k-q)_\nu - (k^2+m^2)\,\eta_{\mu\nu}}{((k+p)^2+m^2)((k-q)^2+m^2)(k^2+m^2)} , \tag{6}$$

where an overall factor of $\mathrm{i}(2\pi)^n$ has been extracted. This amplitude should still be contracted with the photon polarization vectors $\varepsilon^\mu(p)$ and $\varepsilon^\nu(q)$. The photons are on-shell, i.e. $p^2 = q^2 = 0$.

e) What is the degree of divergence of expression (6)? Argue that the final expression should be finite.

f) Show that (6) vanishes upon contraction with p^μ and with q^ν, and explain why this must be the case.

g) **(Bonus)** Defining $P = p - q$ and $Q = p + q$, argue that the most general decomposition of $M_{\mu\nu}$ is

$$M_{\mu\nu} = M_1 \eta_{\mu\nu} + M_2 P_\mu P_\nu + M_3 Q_\mu Q_\nu . \tag{7}$$

Contracting both sides with Q^μ derive the following expression in terms of vector integrals

$$(M_1 + Q^2 M_3)\, Q_\nu = \frac{2\,ge^2}{\mathrm{i}(2\pi)^n} \int \mathrm{d}^n k \Big(\frac{q_\nu}{\big((k+\frac{1}{2}q)^2+m^2\big)\big((k-\frac{1}{2}q)^2+m^2\big)}$$
$$+ \frac{p_\nu}{\big((k+\frac{1}{2}p)^2+m^2\big)\big((k-\frac{1}{2}p)^2+m^2\big)} - \frac{(p+q)_\nu}{\big((k+p)^2+m^2\big)\big((k-q)^2+m^2\big)} \Big) . \tag{8}$$

Derive from this an equation for $M_1 + Q^2 M_3$ in terms of scalar integrals.

More relations like this can be derived to solve for $M_{1,2,3}$ and thereby determine the full invariant amplitude.