Final exam, Mathematical Modelling (WISB357)

Tuesday, 30 Jan 2018, 13.30-16.30, BBG 0.23

- Write your name on each page you turn in, and additionally, on the first page, write your student number and the *total number of pages submitted.*
- For each question, motivation your answer.
- You may make use of results from previous subproblems, even if you have been unable to prove them.
- For this exam you are allowed to bring an A4 with notes on both sides. You may not consult solutions to the exercises, nor use a graphical calculator or smart phone.

Problem 1. The relative air speed $v(x)$ (with units m/s) at a height $x > 0$ above the wing of an airplane flying at constant speed V_0 (m/s) is modelled by the differential equation:

$$\rho V_0 \frac{\partial v}{\partial x} + \mu \frac{\partial^2 v}{\partial x^2} = 0,$$

where $\rho > 0$ is the constant density (kg/m^3) and $\mu > 0$ is the viscosity parameter with units $kg/(m \cdot s)$. In a coordinate system fixed to the wing, the boundary conditions are

$$v(0) = 0, \qquad v(L) = V_0,$$

where L is a given height (in m), far enough from the airplane to neglect its influence.

(a) Nondimensionalize the equation and boundary conditions, using L and V_0 to rescale x and v, respectively. Show that you obtain a dimensionless parameter $Re = \rho V_0 L/\mu$, the "Reynolds number".

(b) The Reynolds number is typically very large. Let $\varepsilon = Re^{-1} \ll 1$, and construct a two-term outer expansion for $v(x)$. Use it to satisfy the boundary condition at $x = L$.

(c) Construct a one-term inner expansion.

(d) Use the matching condition to construct a one-term composite solution.

(e) An airplane manufacturer can use a simplified model outside of the "boundary layer" which is defined as the region where $v(x) < 0.99V_0$. How thick is the boundary layer as a function of ε?

Problem 2. Analysis of traffic in a section of highway within a distance π kilometers of a tunnel (i.e. $x \in [-\pi, \pi]$) has shown that density perturbations $\rho(x, t)$ to the otherwise steady flow evolve according to the relation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} J(\rho) = 0, \qquad J(\rho) = \frac{\rho^2}{2}.$$

During a given morning rush hour, the perturbation is observed to be

$$\rho(0, x) = \rho_0(x) = -\sin x.$$

Answer the following questions:

(a) What are the velocity function $v(\rho)$ and wave speed $c(\rho)$ that hold for this perturbation?

(b) Sketch the characteristics and describe how the density perturbation evolves. (*Hint:* Here it is helpful to consider what happens to the characteristics emanating from points x_0 small enough that the approximation $\sin(x_0) \approx x_0$ holds.)

(c) What is the speed of the resulting shock wave?

Problem 3. A manufacturer of bungee cords has developed a new cord for which the elasticity, expressed in terms of Young's modulus, varies with length according to $E(A) = (A/\ell_0)^{-2}$ for a cord of length ℓ_0, cross-sectional area σ and constant density R_0. To a good approximation, the bungee cord is linearly elastic $T(A) = E(A)\partial U/\partial A$ where $U(A,t)$ is the displacement function. The momentum equation for the motion of the bungee cord is expressed in material coordinates as:

$$R_0 \frac{\partial^2 U}{\partial t^2} = g R_0 + \frac{\partial T}{\partial A}.$$

Suppose a student of mass $M > 0$ is fastened to the end of the bungee cord at $A = \ell_0$. The other end at $A = 0$ is attached to a high bridge, and the student jumps off and bounces around awhile until he reaches a steady state $\partial U/\partial t \equiv \partial^2 U/\partial t^2 \equiv 0$ (due to air friction, apparently).

(a) State the boundary condition that holds for the stress $T(A)$ at $A = \ell_0$ and solve the differential equation for the stress along the cord $T(A)$, $0 \leq A \leq \ell_0$.

(b) State the boundary condition on the displacement $U(A)$ at $A = 0$ and solve for $U(A)$ and the equilibrium length $\ell = \ell_0 + U(\ell_0)$.

(c) Note that $E(A)$ becomes unbounded as $A \to 0$. If the stress is to be finite at $A = 0$, what additional boundary condition should hold on the displacement at $A = 0$? Does your solution satisfy this condition? What does this condition imply about the stiffness or stretchability of the new bungee cord near $A = 0$?