Grading scheme for retake-examination

Exercise 1. 3 points

Part	1.	2.
Points	$1\frac{1}{2}$	$1\frac{1}{2}$

Exercise 2 4 points

Part	1. a)	b)	2.
Points	$1\frac{1}{2}$	1	$1\frac{1}{2}$

Exercise 3. 3 points

Solutions to Exercise (1)

1.) the restriction map

$$Q : \mathcal{C}^1_W(S^1,\mathbb{R}) \longrightarrow \mathcal{C}^1_W(S^1\setminus\{(1,0)\},\mathbb{R})$$
$$f \longmapsto f|_{S^1\setminus\{(1,0)\}}$$

is not open.

Proof:

claim: $Q(\mathcal{C}^1_W(S^1,\mathbb{R}))$ is NOT open in $\mathcal{C}^1_W(S^1\setminus\{(1,0)\},\mathbb{R})$

- Proof: take $f : S^1\setminus\{(1,0)\} \longrightarrow \mathbb{R}$, $x \longmapsto 0$, $f \in Q(\mathcal{C}^1_W(S^1,\mathbb{R}))$

every open nghd of $f \in \mathcal{C}^1_W(S^1\setminus\{(1,0)\},\mathbb{R})$ ~~is of~~ contains a nghd of the form

$$\bigcap_{i=1}^{N} \mathcal{N}^1(f, \underbrace{(U_i,\varphi_i)}_{\text{charts of } S^1\setminus\{(1,0)\}}, \cancel{(V_i,\psi_i)}, K_i, \varepsilon_i)$$

charts of $S^1\setminus\{(1,0)\}$, $K_i \subset U_i$ compact

- claim: $\exists\, h \in \bigcap_{i=1}^{N} \mathcal{N}^1(f,(U_i,\varphi_i),K_i,\varepsilon_i)$ not in the image of $Q(\mathcal{C}^1_W(S^1,\mathbb{R}))$

Proof: take f, $K := \bigcup_{i=1}^{N} K_i \subset S^1$ closed, does not contain $(1,0)$ $\overset{S^1\setminus K \text{ open}}{\Longrightarrow}$

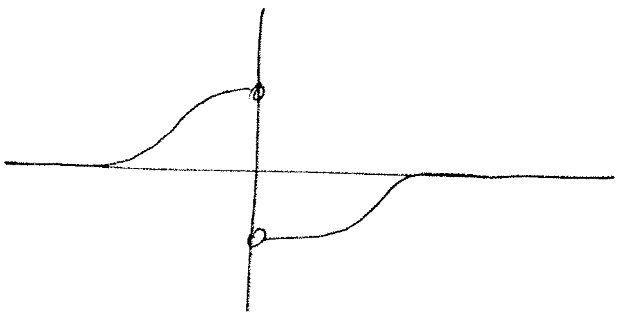
$\exists$ open nghd U of $(1,0) \in S^1$ s.t.

$$K \cap \overline{U} = \phi$$

without loss of generality $U \xrightarrow{\varphi} \mathbb{R}$

$(1,0) \longmapsto 0$

<u>def</u>: $g : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ ~~bump~~ function as follows



$$h := \begin{cases} g \circ \varphi & \text{on } U \\ 0 & \text{on } \overline{U} \end{cases} \quad \text{is } C^1 \text{ and}$$

$$h \in \bigcap_{i=1}^{N} W^1(f,(U_i,\varphi_i),K_i,\varepsilon_i) \quad \text{but} \quad h \notin Q(C^1_W(S^1,\mathbb{R}))$$

$\square$

2.) $U \subset \mathbb{R}^n$ open subset, restriction map

$$R : C^1_S(\mathbb{R}^n,\mathbb{R}) \longrightarrow C^1_S(U,\mathbb{R})$$
$$f \longmapsto f|_U$$

<u>claim</u>: $R(C^1_S(\mathbb{R}^n,\mathbb{R})) \subset C^1_S(U,\mathbb{R})$ is open

<u>Proof</u>: pick $f|_U$, we know:

$\exists$ nghd ~~f to~~ $\mathcal{U}$ of $f \in C^1_S(U,\mathbb{R})$ s.t.

$$\forall g \in \mathcal{U} : \quad T(g) = \begin{cases} g & \text{on } U \\ f & \text{on } \mathbb{R}^n \setminus U \end{cases} \quad \text{is } C^1$$

$\rightarrow$ ~~R~~ $\mathcal{U} \subset R(C^1_S(\mathbb{R}^n,\mathbb{R})) \Longrightarrow$

$R(C^1_S(\mathbb{R}^n,\mathbb{R})) \subset C^1_S(U,\mathbb{R})$ open

Solution to Exercise (2)

1.) $f: M \xrightarrow{C^\infty} M$, $x \in M$ <u>fixed point</u> of f $\iff$ $f(x) = x$

<u>Lefschetz</u> $\iff$ $d_x f$ does not have $+1$ as eigenvalue

consider $(id, f): M \longrightarrow M \times M$

$$\Delta_M = \{(x,x) : x \in M\} \hookrightarrow M \times M$$

<u>observe</u>: x fixed point of f $\iff$ $(id, f)(x) \in \Delta_M$

<u>claim</u>: all fixed points are Lefschetz $\Longrightarrow$ f has only finitely many fixed points

<u>Proof</u>: <u>claim</u>: x fixed point of f is Lefschetz $\iff$

$$d_x(id, f)(T_x M) + T_{(x,x)}\Delta_M = T_x M \times T_x M \quad (*)$$

<u>Proof</u>: $(*)$ holds $\iff$ $\psi: T_x M \xrightarrow{d_x F} T_x M \times T_x M \longrightarrow T_x M$

$$(v, w) \longmapsto v - w$$

$$v \longmapsto v - d_x f(v)$$

is onto $\underset{\text{dim.}}{\iff}$ ψ is injective

i.e. $\nexists v \neq 0: v - d_x f(v) = 0 \iff (d_x f)(v) = v$

$\iff$ $+1$ is no eigenvalue of $d_x f$

<u>hence</u>: all fixed points of f are Lefschetz $\iff$

$(id, f): M \longrightarrow M \times M$ is transversal to Δ_M

$\Rightarrow$ $\underbrace{\{\text{fixed points of } f\}}_{\text{submfd of dim } 0}$ $= (id, f)^{-1}(\Delta_M)$ submfd of codim n

$\overset{M \text{ compact}}{\Longrightarrow}$

$\{\text{fixed points of } f\}$ finite collection of points $\square$

(b) $\mathcal{L} = \{ f : M \xrightarrow{C^\infty} M,$ all fixed points of f are lefschetz$\}$

$= \{ f : M \xrightarrow{C^\infty} M. \ (id, f) \pitchfork \Delta_M \}$

$$\Psi : C^\infty_S(M,M) \longrightarrow C^\infty_S(M, M\times M) \overset{\text{homeomorphism}}{\cong} C^\infty_S(M,M) \times C^\infty_S(M,M)$$

$$f \longmapsto F(id, f) \qquad \text{continuous}$$

consider $\hat{\mathcal{L}} = \{ F : M \longrightarrow M\times M,\ F \pitchfork \underset{\text{closed}}{\underline{\Delta_M}} \}$ open, dense subset of $C^\infty_S(M, M\times M)$

$\mathcal{L} = \Psi^{-1}(\hat{\mathcal{L}}) \quad \Longrightarrow \quad \mathcal{L}$ is open

<u>claim:</u> $\mathcal{L}$ is dense:

consider $f : M \xrightarrow{C^\infty} M$, $(id, f) : M \longrightarrow M\times M$

$\Longrightarrow$ for every nghd $\mathcal{U}$ of id and $\mathcal{V}$ of f in $C^\infty_S(M,M)$,

$\exists F \in \mathcal{U} \times \mathcal{V}$ such that $F \pitchfork \Delta_M$

given $\mathcal{V}$, find $\mathcal{U}$ & $\mathcal{V}'$ nghds of ~~f~~ id & f

sufficiently small s.t. $\mathcal{U} \times \mathcal{V}' \longrightarrow C^\infty_S(M,M)$

$$(\phi, g) \longmapsto g \circ \phi^{-1}$$

has image in $\mathcal{V}$

this is possible by continuity of composition & inversion

given $\underset{(\phi, g)}{F} \in \mathcal{U} \times \mathcal{V}'$ with $F \pitchfork \Delta_M$, consider $f' = \cancel{g \circ \phi^{-1}}\ \phi^{-1} \circ g$

claim: $(id, f'): M \longrightarrow M\times M$ transverse to Δ_M

Proof: $(\phi, g): M \longrightarrow M\times M$ transverse to $\Delta_M \implies$

$\underbrace{(\phi^{-1}; \phi^{-1})\circ(\phi, g)}_{(id,\, \phi^{-1}\circ g)} : M \to M\times M$ transverse to $(\phi^{-1};\phi^{-1})(\Delta_M) = \Delta_M$

□

$\implies$ every nghd of $f \in C^\infty_s(M,M)$ contains $g: M \longrightarrow M$ lefschetz

□

2. claim: $f: S^n \to S^n$ of deg $0 \implies$ f has a fixed point

Proof: $f: S^n \to S^n$ of deg $0 \implies$ f homotopic to constant map

$$g: S^n \to S^n$$
$$x \longmapsto c$$

claim: g is lefschetz:

$\{\text{fixed points of } g\} = \{c\}$

$d_x g = 0$, does not have $+1$ as eigenvalue

$\implies L(g) = 1$

Suppose $f: S^n \to S^n$ does not have any fixed point $\implies$

f is lefschetz & $L(f) = 0$

but $f \sim g \implies L(g) = L(f)$ ↯

□

Solutions to Exercise (3)

$M \subset \mathbb{R}^{q+1}$ ~~compact~~, $v \in S^q$ def $f_v : M \longrightarrow \mathbb{R}$
$x \longmapsto \langle v, x \rangle$

~~claim~~

consider
$$S^q \times M \xrightarrow{d_M F} T^*M \qquad \text{smooth}$$
$$(v, m) \longmapsto d_m(f_v)$$

equals
$$S^q \times M \longrightarrow T^*\mathbb{R}^{q+1}|_M \cong M \times (\mathbb{R}^{q+1})^* \longrightarrow T^*M \qquad \text{smooth}$$
$$(v, m) \longmapsto (m, \langle v, - \rangle)$$

claim: $d_M F \pitchfork Z = \{(m,0) \subset T^*M\} \subset T^*M$

Proof:

- $(d_M F)(v, m) \in Z \iff v \perp T_m M$

Proof: $S^q \times M \longrightarrow M \times (\mathbb{R}^{q+1})^* \longrightarrow T^*M$ maps to $(m, 0)$
(v, m)
$\iff$ restriction of $\langle v, - \rangle$ to $T_m M$ vanishes
$\iff v \perp T_m M$ $\square$

- suppose $v \perp T_m M \Longrightarrow$

$$d_{(v,m)}(d_M F) : T_v S^q \times T_m M \longrightarrow T_{(m,0)} T^*M \cong T_m M \oplus T_m^* M$$
$$(w, \xi) \longmapsto (\xi, \langle w, - \rangle)$$

$w \in T_v S^q \iff$ ~~$v \perp w$~~

~~image~~

$$d_{(v,m)}(d_H F)(T_v S^q \times T_m M) + T_m M = T_{(m,0)} T^* M \iff$$

$$\begin{aligned} T_v S^q &\longrightarrow T_m^* M \\ \omega &\longmapsto \langle \omega, - \rangle \end{aligned}$$ is surjective

$$\begin{aligned} \mathbb{R}^{q+1} &\twoheadrightarrow T_m^* M \\ \omega &\longmapsto \langle \omega, - \rangle \end{aligned}$$ is surjective

$$\mathbb{R}^{q+1} = T_v S^q \oplus \underbrace{\langle v \rangle}_{\text{maps to zero}} \longmapsto T_m^* M \implies T_v S^q \to T_m^* M \text{ is surjective}$$

$\square$

- $\Rightarrow$ $d_H F \pitchfork Z$

parametricT $\implies$ $\{v \in S^q : \underbrace{df_v \pitchfork Z}\}$ dense

$\Updownarrow$

f_v Morse

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