Cie$^\infty$

MOAWOA
2021-2022

A - E s k w a d r a a t

- You have from 13:00 until 17:00 to solve the problems, so four hours total.
- Raise your hand if you have a question.
- Calculators are not allowed.
- You can score between 0 and 10 points on each problem.
- Be explicit when you use a theorem. Give sources for obscure theorems.
- Use a separate sheet for each problem.
- Clearly write DRAFT on any draft sheet.
- You can write your solutions in English or in Dutch.

Opgave 1 Determine a natural number n such that for all $k = 1, 2, \ldots, 9$ the number kn contains all digits $0, 1, \ldots, 9$ at least once in its decimal expansion. You do not need to justify your answer. Your score is dependent on the number of digits in your number. Each value of $k = 1, \ldots, 9$ for which kn does not contain all digits $0, 1, \ldots, 9$ loses you 1 point. The participant with the smallest number gets 1 bonus point.

Number of digits	$100-300$	$60-99$	$40-59$	$30-39$	≤ 29
Score	5	6	7	8	9

Opgave 2 Let $n \in \mathbb{N}$. Show that there exists a value $\alpha_n \in (0, \infty)$ with the following property. For each $x_1, \ldots, x_k \in [0,1]^n$ there exists $x \in [0,1]^n$ such that the average distance from x to x_i for $i = 1, \ldots, k$ is equal to α_n.

Opgave 3 There are $n+1$ bins in a row with s stones in the first bin. In a move, you chose a bin with $b \geq 1$ stones and you move one of the stones at most b bins. The goal is to get all the stones in the last bin.
(a) [5pt] Show that you need at least $\frac{n}{1} + \frac{n}{2} + \ldots + \frac{n}{s}$ moves.
(b) [5pt] For $s = k(k+1)/2$ show that $s + (n-1)k$ moves suffices.

Opgave 4 For which strictly monotonically increasing $f : \mathbb{N} \to \mathbb{N}$ does there exist a strictly monotonically increasing $g : \mathbb{N} \to \mathbb{N}$ with the following two properties?
(i) For all $k \geq 0$ we have for n large enough that $f^k(n) \leq g(n)$.[1]
(ii) The value $f(g(n)) - g(f(n))$ can become arbitrarily large.

Opgave 5 Let $c > 1$. Show that there exists $r > 1$ such that $\lim_{n\to\infty} {}^{n+1}r / {}^{n}c \in (0, \infty)$.[2]

Opgave 6 For $k \geq 1$, define $f_k : \mathbb{N} \to \mathbb{N}$ by $f_k(n) = 1^k + 2^k + \ldots + n^k$. Determine for which prime powers p^a and for which $k \geq 1$ the function f_k is surjective modulo p^a.

[1] We write $f^k(n)$ for the result after $k \geq 0$ applications of f on n.
[2] We write ${}^{n}x = x^{{}^{n-1}x}$ with the basis ${}^{0}x = 1$. So ${}^{n}x = x \uparrow\uparrow n$ is tetration, repeated exponentiation.