MOAWOA
2022-2023

- You have from 13:00 until 17:00 to solve the problems, so four hours total.
- Raise your hand if you have a question.
- Calculators are not allowed.
- You can score between 0 and 10 points on each problem.
- Be explicit when you use a theorem. Give sources for obscure theorems.
- Use a separate sheet for each problem.
- Clearly write DRAFT on any draft sheet.
- You can write your solutions in English or in Dutch.

Problem 1 Let $n \geq 2$ be an integer. Given positive real numbers $x_1, x_2, \ldots, x_n$ such that

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \leq \left(n + \frac{4}{3}\right)^2,$$

show that $\max(x_1, x_2, \ldots, x_n) \leq 9 \min(x_1, x_2, \ldots, x_n)$.

Problem 2 Alicia and Bobaniel play the following game, taking alternating turns, with Alicia playing first. They have a square grid of lamps with side length n, each of which can be switched on and off. Initially all lamps are off. At each turn, a player either

- switches on a currently off lamp, or
- switches off a currently on lamp and switches on the closest lamps to the left and to the right (if such a lamp exists, for each separately), or
- switches off a currently on lamp and switches on the closest lamps to the top and to the bottom (if such a lamp exist, for each separately).

A player loses when they return the lamps collectively to a state they have been in before. Which player has a winning strategy?

Problem 3 Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing function with $\lim_{x\to\infty} f(x) = 1$. Show that the integral

$$\int_0^\infty \frac{f(x) - f(x+1)}{\ln(f(x))} \mathrm{d}x$$

diverges.
[*Hint: Show that if the integral were to converge,* $\lim_{x\to\infty} \frac{f(x+1)-1}{f(x)-1} = 0$.]

Problem 4 Fix any prime number p and consider the function $f : \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}$ such that for all $n \in \mathbb{N}_{\geq 0},\ a \in \{1, \ldots, p-1\}$ and $b \in \{0, \ldots, p-1\}$:

$$f(b) = b,$$

$$f(p \cdot n) = f(n),$$

$$f(p^2 \cdot n + p \cdot b + a) = p \cdot f(p \cdot n + a) + f(p \cdot n + b) - p \cdot f(n).$$

Determine $\sum_{k=p^{2022}}^{p^{2023}} f(k)$.

Problem 5 Consider $\{0,1\}^n$ with termwise addition $(+)$ and multiplication $(\times)$ modulo 2. For $i \in \{1, \ldots, n\}$, let $e_i = (0, \ldots, 1, \ldots, 0) \in \{0,1\}^n$ be the element with only nonzero coordinate i.

Let $\mathcal{S}$ be a nonempty subset of $\{0,1\}^n$ such that:

(i) if $x, y \in \mathcal{S}$, then $x \times y \in \mathcal{S}$ and $x + y + x \times y \in \mathcal{S}$, and

(ii) if $x \in \mathcal{S}$ and $x \neq \mathbf{0} = (0, \ldots, 0)$, then there is an element $z \in \mathcal{S}$ such that $x \times z = z$ and $z + x = e_i$ for some $i \in \{1, 2, \ldots, n\}$.

Suppose that $f : \mathcal{S} \to \mathbb{R}_{\geq 0}$ is a function such that for $x, y \in \mathcal{S}$

$$f(x + y + x \times y) = f(x) + f(y) - f(x \times y).$$

Show that there are $f_1, \ldots, f_n \in \mathbb{R}_{\geq 0}$ such that for all $x \in \mathcal{S}$,

$$f(x) = \sum_{i\,:\,x \times e_i \neq 0} f_i.$$

[*Hint: Show that if* $x + y = e_i$, $|f(x) - f(y)| = f_i$.]

Problem 6 Let $n \geq 4$ be an integer and consider n lighthouses $L_1, L_2, \ldots, L_n$ which are placed uniform randomly and independently around a circular lake. Suppose that each lighthouse lights up a continuous quarter of the sky and can rotate parallel to the ground. What is the probability that some lighthouse can light up all other lighthouses? (I.e. can be rotated such that its light hits all other lighthouses.)