MOAWOA
2022-2023

- You have from 13:00 until 17:00 to solve the problems, so four hours total.
- Raise your hand if you have a question.
- Calculators are not allowed.
- You can score between 0 and 10 points on each problem.
- Be explicit when you use a theorem. Give sources for obscure theorems.
- Use a separate sheet for each problem.
- Clearly write DRAFT on any draft sheet.
- You can write your solutions in English or in Dutch.

Problem 1 Let $n \geq 2$ be an integer. Given positive real numbers $x_1, x_2, \ldots, x_n$ such that

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \leq \left(n + \frac{4}{3}\right)^2,$$

show that $\max(x_1, x_2, \ldots, x_n) \leq 9 \min(x_1, x_2, \ldots, x_n)$.

Uitwerking:

Problem 2 Alicia and Bobaniel play the following game, taking alternating turns, with Alicia playing first. They have a square grid of lamps with side length n, each of which can be switched on and off. Initially all lamps are off. At each turn, a player either

- switches on a currently off lamp, or
- switches off a currently on lamp and switches on the closest lamps to the left and to the right (if such a lamp exists, for each separately), or
- switches off a currently on lamp and switches on the closest lamps to the top and to the bottom (if such a lamp exist, for each separately).

A player loses when they return the lamps collectively to a state they have been in before. Which player has a winning strategy?

Uitwerking: A winning strategy for Alicia is to keep the board 180° rotationally symmetric and the middle lamp turned on. Whatever Bobaniel ("Bobby" for short) does on his turn, it remains possible for Alicia to return to such a state. Eventually there will only be

Problem 3 Let $f:[0,\infty)\to\mathbb{R}$ be a strictly decreasing function with $\lim_{x\to\infty} f(x)=1$. Show that the integral

$$\int_0^\infty \frac{f(x)-f(x+1)}{\ln(f(x))}\mathrm{d}x$$

diverges.
[*Hint: Show that if the integral were to converge,* $\lim_{x\to\infty}\frac{f(x+1)-1}{f(x)-1}=0$.]

Uitwerking: Since

$$\frac{f(x)-f(x+1)}{\ln(f(x))} > \frac{f(x)-f(x+1)}{f(x)-1},$$

it is sufficient to prove that

$$\int_0^\infty \frac{g(x)-g(x+1)}{g(x)}dx$$

diverges for $g=f-1$. The convergence of this integral would imply $\frac{g(x+1)}{g(x)}\to 1$, but by

$$0<\frac{g(a+1)}{g(a)}<\int_a^{a+1}\frac{g(x)}{g(a)}dx=\int_a^\infty \frac{g(x)-g(x+1)}{g(a)}dx\le\int_a^\infty \frac{g(x)-g(x+1)}{g(x)}dx\to 0$$

that $\frac{g(x+1)}{g(x)}\to 0$. Thus this integral diverges, and so does the first one.

Problem 4 Fix any prime number p and consider the function $f:\mathbb{N}_{\ge 0}\to\mathbb{N}_{\ge 0}$ such that for all $n\in\mathbb{N}_{\ge 0},\ a\in\{1,\dots,p-1\}$ and $b\in\{0,\dots,p-1\}$:

$$f(b)=b,$$

$$f(p\cdot n)=f(n),$$

$$f(p^2\cdot n+p\cdot b+a)=p\cdot f(p\cdot n+a)+f(p\cdot n+b)-p\cdot f(n).$$

Determine $\sum_{k=p^{2022}}^{p^{2023}} f(k)$.

Uitwerking: The function f simply flips the base-p representation of its input. Clearly for (zero- and) one-digit numbers this holds, and for an n-digit number it can be derived inductively from the number with the last two digits removed.
Using this identity, you can derive

$$f(\{p^{2022},\dots,p^{2023}-1\})=\{k\in\mathbb{N}|k<p^{2023},k\not\equiv 0 \pmod p\},$$

and thus

$$\sum_{k=p^{2022}}^{p^{2023}} f(k)=1+\sum_{k=0}^{p^{2023}-1}k-p\cdot\sum_{k=0}^{p^{2022}-1}k=1=1+\frac{(p^{2023}-1)(p^{2023})}{2}-p\frac{(p^{2022}-1)(p^{2022})}{2}=$$

$$1+p^{2023}\frac{p^{2023}-p^{2022}}{2}=1+p^{4045}\frac{p-1}{2}.$$

Problem 5 Consider $\{0,1\}^n$ with termwise addition $(+)$ and multiplication $(\times)$ modulo 2. For $i \in \{1,\dots,n\}$, let $e_i = (0,\dots,1,\dots,0) \in \{0,1\}^n$ be the element with only nonzero coordinate i.

Let $\mathcal{S}$ be a nonempty subset of $\{0,1\}^n$ such that:

(i) if $x, y \in \mathcal{S}$, then $x \times y \in \mathcal{S}$ and $x + y + x \times y \in \mathcal{S}$, and

(ii) if $x \in \mathcal{S}$ and $x \neq \mathbf{0} = (0,\dots,0)$, then there is an element $z \in \mathcal{S}$ such that $x \times z = z$ and $z + x = e_i$ for some $i \in \{1,2,\dots,n\}$.

Suppose that $f : \mathcal{S} \to \mathbb{R}_{\geq 0}$ is a function such that for $x, y \in \mathcal{S}$

$$f(x + y + x \times y) = f(x) + f(y) - f(x \times y).$$

Show that there are $f_1,\dots,f_n \in \mathbb{R}_{\geq 0}$ such that for all $x \in \mathcal{S}$,

$$f(x) = \sum_{i\,:\,x \times e_i \neq 0} f_i.$$

[*Hint: Show that if* $x + y = e_i$, $|f(x) - f(y)| = f_i$.]

Uitwerking: By identifying each $x \in \{0,1\}^n$ with the set $S_x = \{k \in \{0,\dots,n\} | x_k \neq 0\}$, you get $\{0,1\}^n = \mathcal{P}(\{0,\dots,n\})$, $e_i = \{i\}$, and the operations $\times$ and $(x,y) \mapsto x+y+x\times y$ correspond to intersection and union respectively. The second requirement turns into $S_z \subset S_x$ and $S_x \setminus S_x = \{i\}$.
Suppose $S_y \subset S_x, S_w \subset S_z$ and $S_x \setminus S_y = S_z \setminus S_w = \{i\}$. Then

$$f(S_y \cup S_z \cup \{i\}) = f(S_x) + f(S_w) - f(S_y \cup S_w) = f(S_y) + f(S_z) - f(S_y \cup S_w)$$

and thus $f(S_x) - f(S_y) = f(S_z) - f(S_w) = f_i$. Now for each $x \in \mathcal{S}$ there are $\mathbf{0} = x_1, \cdots, x_k = x \in \mathcal{S}$ s.t. $S_{x_i} \subset S_{x_{i+1}}$ and $S_{x_{i+1}} \setminus S_{x_i} = \{v_i\}$, so

$$f(x_{i+1}) = f(x_i) + f_i$$

and

$$f(x) = \sum_{i \in S_x} f_i = \sum_{i\,:\,x \times e_i} f_i.$$

Problem 6 Let $n \geq 4$ be an integer and consider n lighthouses $L_1, L_2, \ldots, L_n$ which are placed uniform randomly and independently around a circular lake. Suppose that each lighthouse lights up a continuous quarter of the sky and can rotate parallel to the ground. What is the probability that some lighthouse can light up all other lighthouses? (I.e. can be rotated such that its light hits all other lighthouses.)

Uitwerking: The problem can be restated as such: There are n points placed uniformly randomly, independently on a circle. What is the probability that there is a half-circle which contains all but one point? (Since the angle between three consecutive points is $< 90\,\text{deg}$, i.e. acute, iff the first and last point lie in a half-circle not containing the middle point.)
There are at most 2 acute angles for $n \geq 4$ (draw a picture), which (if there are 2) are adjacent. Thus there is at most 1 acute point followed clockwise by 2 obtuse points, say P_1, P_2, P_3. For P, Q the clockwise arc from P to Q, this happens iff $P_3, \ldots, P_n \in [P_2, -P_2]$ and $P_1 \notin [P_3, -P_3]$. If the arc $[P_1, P_2]$ has length x, the probability that P_1, P_2 are the relevant points is

$$\int_0^{1/2} (\frac{1}{2})^{n-2} - x^{n-2} dx = \frac{n-2}{}$$