- There are 4 hours available for the problems.
- Each problem is worth 10 points.
- Be clear when using a theorem. When you are using an obscure theorem, cite a source.
- Use a different sheet for each problem.
- Clearly write DRAFT on any draft page you hand in.

MOAWOA :: SOLUTIONS

May 4, 2018

Problem 1. Determine all sequences $(a_1, a_2, \ldots, a_{2018})$ of positive integers such that

(i) $a_1 + a_2 + \ldots + a_{2018} = 3 \cdot 2018$;

(ii) the sum of consecutive a_i is never a power of 2. (In particular, none of the a_i is a power of 2.)

(A power of 2 is a number of the form 2^k with $k \geq 1$ an integer.)

Proposed by Merlijn Staps.

Solution. There are three sequences possible: $(3, 3, \ldots, 3)$, $(1, 5, 1, 5, \ldots, 1, 5)$, and $(5, 1, 5, 1, \ldots, 5, 1)$. We first show that these are valid. Clearly all three sequences satisfy (i). The first sequence satisfies (ii) because the sum of consecutive terms is always a multiple of 3. For the other two sequences (ii) follows from the fact that the sum of consecutive terms is either odd (if we sum an odd number of terms) or a multiple of 3 (if we sum an even number of terms), hence never a power of 2.
We now prove that there can be no other valid sequences. First, we note that $a_{2i-1} + a_{2i}$ cannot be equal to 2 (a power of 2), 3 (then one of them would be 2, hence a power of 2), 4 (a power of 2) or 5 (then one of them would be 2 or 4, hence a power of 2). It follows that $a_{2i-1} + a_{2i} \geq 6$ for all i. By summing this inequality we obtain

$$3 \cdot 2018 = \sum_{i=1}^{2018} a_i = \sum_{i=1}^{1009} (a_{2i-1} + a_{2i}) \geq \sum_{i=1}^{1009} 6 = 6 \cdot 1009,$$

meaning that there should be equality in each inequality. We conclude that $a_{2i-1} + a_{2i} = 6$ for $i = 1, 2, \ldots, 1009$, which implies that (a_{2i-1}, a_{2i}) should be $(3, 3)$, $(1, 5)$, or $(5, 1)$. Suppose the pair $(3, 3)$ occurs at least once. Because a 3 cannot occur adjacent to a 1 or a 5 in the sequence (both $3 + 1$ and $3 + 5$ are powers of 2), it then follows that $a_i = 3$ for all i (the first solution). If not, each pair is equal to $(1, 5)$ or $(5, 1)$. Because there cannot be two consecutive ones in the sequence, the sequence must be of the form

$$\underbrace{1, 5, \quad 1, 5, \quad \ldots \quad 1, 5,}_{p} \quad \underbrace{5, 1, \quad 5, 1, \quad \ldots \quad 5, 1}_{q}$$

where $p + q = 1009$. If p and q are both positive then the sequence contains either $5, 5, 1, 5$ (if $q \geq 2$) or $5, 1, 5, 5$ (if $p \geq 2$) so then there are consecutive terms summing to 16. This contradiction shows that we must have $p = 0$ or $q = 0$, leaving us with the second and third solution. □

Problem 2. Let $k > 1$ be an integer. We list all k-element subsets of $\{1, 2, \dots, 2k-1\}$ and in each of these subsets we color one element red and one (not necessarily distinct) element blue. We say our assignment of colors is nice if whenever A and B are subsets among our list with $|A \cap B| = \ell$, the red element in A differs from the blue element in B. Does there always exists a nice assignment of colors

(a) if $\ell = 1$?

(b) if $\ell = 2$?

Proposed by Stijn Cambie.

Solution.

(a) Yes. In each set A we color the minimal i such that A contains both i and $i+1$ in both red and blue (where we read $i+1$ as 1 if $i = 2k-1$). Note that such an i always exists, because A contains more than half of the elements of $\{1, 2, \dots, 2k-1\}$. If the red element in A equals the blue element in B (say both equal to i), then both A and B contain i and $i+1$, so we cannot have $|A \cap B| = 1$. □

(b) Yes. In each set we color the smallest element red and the largest element blue. Suppose $|A \cap B| = 2$; then there are $a < b$ that are contained in both A and B. Now the red element in A is at most a, whereas the blue element in B is at least b, so these numbers cannot be equal. □

Problem 3. A real $n \times n$-matrix $A = (A_{ij})_{i,j=1}^n$ satisfies $A_{ii} = 1$ for $1 \le i \le n$ and $A_{ij} + A_{ji} = 1$ for $1 \le i < j \le n$. Show that $\det A > 0$.

Proposed by Daniël Kroes.

Solution. Note that the determinant of A equals the product of its eigenvalues. Since the non-real eigenvalues come in pairs $(\lambda, \overline{\lambda})$ whose product is $|\lambda|^2 > 0$, it suffices to show that the product of the real eigenvalues of A is positive. We will show that in fact every real eigenvalue is positive.
Let r be a real eigenvalue of A with corresponding eigenvector $v \neq 0$. Note that $A + A^T = J_n + I_n$, where J_n is an $n \times n$-matrix with all entries equal to 1. We obtain

$$v^T(A + A^T)v = v^T J_n v + v^T I_n v = (v^T j)^2 + |v|^2 > 0$$

where $j = (1, 1, \dots, 1)^t$. On the other hand, we have

$$v^T(A + A^T)v = v^T A v + v^T A^T v = v^T r v + (rv)^T v = 2r|v|^2$$

Since $|v|^2$ is positive, it follows that r is positive as well. □

Problem 4. Does there exist a smooth function $f : \mathbb{R} \to \mathbb{R}$ with the property that for all non-negative integers n the number of roots of $f^{(n+1)}$ is (strictly) greater than the number of roots of $f^{(n)}$?

Proposed by Leslie Molag.

Solution. The answer is yes. An example is given by $f(x) = e^{p(x)}$, where p is a polynomial with even degree and negative leading coefficient. In this case $f^{(n)} = q_n f$, where (q_n) is a sequence of polynomials of strictly increasing degrees. By the mean value theorem $f^{(n+1)}$ has a root between any two consecutive roots of $f^{(n)}$. Hence the number of roots of $f^{(n+1)}$ is at least one less then the number of roots of $f^{(n)}$. Furthermore, note that by our choice of p we know that f and any of its derivatives tend to 0 as $|x| \to \infty$. Denote by a and b the smallest and largest root of $f^{(n)}$ respectively. Since $f^{(n)}$ approaches 0 as $|x| \to \infty$ it must attain extremal values at some points $A \in (-\infty, a]$ and $B \in [b, \infty)$. This implies that $f^{(n+1)}(A) = f^{(n+1)}(B) = 0$ and we conclude that $f^{(n+1)}$ has strictly more roots than $f^{(n)}$. $\square$

Remark. For $p(x) = -x^2/2$ we obtain the Hermite polynomials. In this case $f^{(n)}$ has exactly n roots.

Problem 5. We sample a random permutation σ of the numbers 1, 2, …, n, uniformly from the set of all $n!$ permutations. For a set $A \subset \{1, 2, \ldots, n\}$ we define the event

$$X_A = \{\text{ all elements of } A \text{ belong to the same cycle of } \sigma \}.$$

Show that for any two sets S and T with at least 2 elements, the events X_S and X_T are positively correlated.

Proposed by Harry Smit and Merlijn Staps.

Solution. We first show that $\mathbb{P}(X_A) = |A|^{-1}$ for any A. Without loss of generality we assume that A is of the form $\{1, 2, \ldots, k\}$. A random permutation σ of 1, 2, …, n can be constructed as follows:

- Choose $\sigma(1)$ uniformly from $\{1, 2, \ldots, n\}$, then choose $\sigma(\sigma(1))$ uniformly from the remaining numbers, then choose $\sigma(\sigma(\sigma(1)))$ uniformly from the remaning numbers, etcetera. Continue until 1 is chosen and the cycle containing 1 is determined.
- Choose a random permutation of the remaining numbers.

The probability that 1, 2, …, k are all in the same cycle (the cycle containing 1), is the probability that in the first step we select each of 2, 3, …, k before we select 1. By symmetry, this happens with probability $k^{-1} = |A|^{-1}$.
We now turn to the problem statement. Note that $X_S \cap X_T \supset X_{S \cup T}$, hence

$$\mathbb{P}(X_S \cap X_T) \geq \mathbb{P}(X_{S\cup T}) = \frac{1}{|S \cup T|} \geq \frac{1}{|S| + |T|} \geq \frac{1}{|S||T|} = \mathbb{P}(X_S)\mathbb{P}(X_T),$$

where the last inequality follows from $(|S| - 1)(|T| - 1) \geq 1$. It now suffices to show that we cannot have equality everywhere.

- If S and T are disjoint, we have $\mathbb{P}(X_S \cap X_T \setminus X_{S\cup T}) > 0$ because with positive probability there exists a cycle containing all elements of S and another cycle containing all elements of T. This means the first inequality is strict.
- If S and T are not disjoint, we have $|S|+|T| > |S \cup T|$, meaning that the second inequality is strict.

We conclude that at least one of the first two inequalities is strict. This means that $\mathbb{P}(X_S \cap X_T) > \mathbb{P}(X_S)\mathbb{P}(X_T)$, hence X_S and X_T are positively correlated. $\square$

Problem 6. Determine the smallest constant $C > 0$ with the following property: if $n \geq 4$ is a positive integer, then there exist positive integers a, b, c and d such that $a+b+c+d = n$ and $\operatorname{lcm}(a,b,c,d) \leq Cn$.

Proposed by Merlijn Staps.

Solution. The smallest such C is $C = \frac{1}{2}$.
We first show $C = \frac{1}{2}$ has the desired property. If $n = 2k$ with $k \geq 2$ is even we take $(a,b,c,d) = (1,1,k-1,k-1)$ with $a+b+c+d = 2k = n$ and $\operatorname{lcm}(a,b,c,d) = k-1 < k = Cn$. If $n = 4k+1$ we choose $(a,b,c,d) = (1,k,k,2k)$ with $a+b+c+d = 4k+1 = n$ and $\operatorname{lcm}(a,b,c,d) = 2k < \frac{4k+1}{2} = Cn$. Finally, if $n = 4k+3$ we choose $(a,b,c,d) = (1,2,2k,2k)$ with $a+b+c+d = 4k+3 = n$ and $\operatorname{lcm}(a,b,c,d) = 2k < \frac{4k+3}{2} = Cn$. We conclude that $C = \frac{1}{2}$ works (with the inequality being strict).
Next we prove that $C < \frac{1}{2}$ do not have the required property. Take such C and let $N \geq 5$ be a natural number satisfying $C < \frac{1}{2} - \frac{1}{N}$. By Dirichlet's theorem on primes in arithmetic progressions, there exists a prime n such that $n \equiv -1 \mod N!$. We will show that if $a+b+c+d = n$ we always have $\operatorname{lcm}(a,b,c,d) \geq (\frac{1}{2} - \frac{1}{N})n > Cn$. Indeed, let $L = \operatorname{lcm}(a,b,c,d)$. If $L \geq \frac{n}{2}$ then clearly $L \geq (\frac{1}{2} - \frac{1}{N})n$, so suppose $L < \frac{n}{2}$. Without loss of generality we assume that $a \leq b \leq c \leq d$. Then it follows that $d \leq L < \frac{n}{2} \leq \frac{4d}{2} = 2d$, so since L is a multiple of d we must have $L = d$. It follows that a, b and c divide d. From $a \leq b \leq c \leq d < \frac{n}{2}$ we have $c = \frac{d}{2}$ of $c = d$, because otherwise $a+b+c+d < n$. If $c = \frac{d}{2}$ we similarly find $b = \frac{d}{3}$ or $b = \frac{d}{2}$. In the first case, we have $a = \frac{d}{3}$ (hence $13d = 6n$), $a = \frac{d}{4}$ (hence $12n = 25d$), or $a = \frac{d}{5}$ (hence $30n = 61d$). We find that n is divisible by a prime at most 61, which is a contradiction because n is a prime that is at least $5! - 1 = 119$. So $b = \frac{d}{3}$ is impossible. If $b = \frac{d}{2}$ we find $n = 2d + a$ where $a \mid d$. From $a \mid n$ it now follows that $a = 1$ (n is prime). Since d is even (otherwise b would not be an integer) we find $n \equiv 1 \mod 4$, contradiction. We have now excluded $c = \frac{d}{2}$, so the case $c = d$ remains. Note that in this case $\gcd(a,b) = 1$, because $\gcd(a,b)$ divides $n = 2d + a + b$. Therefore we have $ab \mid d$. We claim that $b \leq \frac{2d}{N}$. If not, we would have $b = \frac{d}{k}$ with $2k < N$, and from $ab \leq d$ it follows that $a \leq k$. Now we find $kn = ka + kb + kc + kd = ka + (2k+1)d$, hence $2k+1 \mid k(n-a)$. It follows that $n - a$ is divisible by $2k+1$. However, from $2k+1 \leq N$ we have $2k+1 \mid N! \mid n+1$, hence $a+1$ is divisible by $2k+1$. This contradicts $1 \leq a+1 \leq k+1$. So we must indeed have that $b \leq \frac{2d}{N}$. It follows that $a + b \leq \frac{4d}{N} \leq \frac{2n}{N}$, and

$$L = \operatorname{lcm}(a,b,c,d) = n = \frac{n-a-b}{2} \geq \frac{n - \frac{2n}{N}}{2} = \left(\frac{1}{2} - \frac{1}{N}\right) n,$$

as required. In conclusion, $C < \frac{1}{2}$ is not possible, so $C = \frac{1}{2}$ is the smallest. □