- There are 4 hours available for the problems.
- Each problem is worth 10 points.
- Be clear when using a theorem. When you are using an obscure theorem, cite a source.
- Use a different sheet for each problem.
- Clearly write DRAFT on any draft page you hand in.
- **This is an alternative paper for people who proposed a problem for the actual competition.**

MOPWOP :: SOLUTIONS

May 13, 2016

Problem 1. An invertible 2×2-matrix M with real entries is called a *MOAWOA-matrix* if its inverse M^{-1} can be obtained by permuting the entries of M. Show that if M is a MOAWOA-matrix, then so is M^2.
An invertible 3×3-matrix M with real entries is called a *MOPWOP-matrix* if its inverse M^{-1} can be obtained by permuting the entries of M. Does the same conclusion hold for MOPWOP-matrices?

Proposed by Merlijn Staps and Jeroen Huijben (Universiteit Utrecht).

Solution. We first deal with the MOAWOA-matrices. Write $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let $D = \det M$. Then we have $M^{-1} = \frac{1}{D}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Now, suppose M is a MOAWOA-matrix. Then M^{-1} can be obtained by permuting the entries of M. Therefore, we must have $|a| + |b| + |c| + |d| = |\frac{d}{D}| + |\frac{-b}{D}| + |\frac{-c}{D}| + |\frac{a}{D}| = \frac{|a|+|b|+|c|+|d|}{|D|}$. Since M is invertible, we have $|a|+|b|+|c|+|d| > 0$. It follows that $|D| = 1$, hence $D = \pm 1$. First suppose $D = 1$. Then we have $[a, b, c, d] = [d, -b, -c, a]$, where we use square brackets to denote multisets. We find $[b, c] = [-b, -c]$, which implies that $b = -c$. We therefore have $M = \begin{pmatrix} a & b \\ -b & d \end{pmatrix}$ with $ad + b^2 = 1$. Conversely, any matrix of this form is a MOAWOA-matrix. In particular, since $\det(M^2) = 1$ and $M^2 = \begin{pmatrix} a^2-b^2 & ab+bd \\ -ba-bd & -b^2+d^2 \end{pmatrix}$ the matrix M^2 is a MOAWOA-matrix. Now suppose $D = -1$. Then we have $[a, b, c, d] = [-d, b, c, -a]$. It now follows that $a = -d$. Therefore, $M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. We find $M^2 = \begin{pmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ since $a^2 + bc = -\det M = 1$. Clearly, M^2 is a MOAWOA-matrix.

A similar conclusion does not hold for MOPWOP-matrices. A counterexample is given by

$$M = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad M^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Computing

$$M^2 = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M^{-2} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

shows that whereas M is a MOPWOP-matrix, M^2 is not. □

Problem 2. Suppose I and J are (real) open intervals of finite positive length, each interval not containing the other. Show that there exists a $\lambda \neq 0$ such that $x \mapsto e^{\lambda x}$ maps I and J to intervals of equal length if and only if I and J have different lengths.

Proposed by Leslie Molag (Katholieke Universiteit Leuven).

Solution. Denote the endpoints of I and J by $a < b$ and $c < d$ respectively. Without loss of generality $d > b$ and $c > a$ (since each interval does not contain the other). Let us define the function

$$f(\lambda) = \begin{cases} \frac{d-c}{b-a} & \text{if } \lambda = 0, \\ \frac{e^{d\lambda}-e^{c\lambda}}{e^{b\lambda}-e^{a\lambda}} & \text{otherwise.} \end{cases}$$

Note that I and J being mapped to intervals of equal length by $x \mapsto e^{\lambda x}$ is equivalent to f attaining the value 1 in some $\lambda \neq 0$. The function f is continuous by construction. We notice that $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to 0$ as $x \to -\infty$. When we assume that I and J have distinct lengths we also know that $f(0) \neq 1$, thus by the intermediate value theorem there exists an $\lambda \neq 0$ such that $f(\lambda) = 1$. When I and J have the same length we have $f(\lambda) = e^{(c-a)\lambda}$, which does not equal 1 for any $\lambda \neq 0$. □

Problem 3. Consider n people that stand in a circle. Initially, each of them holds a red and a blue ball. In a turn, each person chooses one of his balls and hands it to the person on his right. Thus, after a turn everyone again holds two balls, but the distribution of colors may have changed. Determine all positive integers n for which there exists a sequence of turns that, from the described starting point, visits all possible color distributions of the $2n$ balls, without any color distribution occurring twice.

Proposed by Wouter Zomervrucht (Freie Universität Berlin).

Solution. Number the people $1, \ldots, n$ in circular order. We denote by $[a_1 a_2 \cdots a_n]$ the color distribution where person i has precisely a_i blue balls (and $2 - a_i$ red balls). A circular permutation of $[a_1 a_2 \cdots a_n]$ is one of the distributions $[a_{r+1} a_{r+2} \cdots a_{r+n}]$ with $1 \leq r \leq n$, taking the indices modulo n. We will sometimes write $(ab)^k$ to denote the k-fold iteration $abab \cdots ab$. We show that it can be done for $n = 1, 3$ only. The case $n = 1$ is clear. For $n = 3$ one can write down an explicit solution, e.g. $[111] \to [012] \to [102] \to [201] \to [210] \to [120] \to [021]$.
Suppose $n = 2k$ is even. From the distribution $[(20)^k]$ one can only go to the initial position $[1^n]$, so if a suitable sequence of turns exists, $[(20)^k]$ must be the final distribution. However, the same applies to $[(02)^k]$, contradiction.
Suppose $n = 2k+1 \geq 5$ is odd and suppose a suitable sequence of turns exists. By symmetry we may assume that no circular permutation Q of $[(20)^k 1]$ is the final distribution. The neighbors of such Q in the turn sequence can only be the initial position P_0 or some circular permutation of $[1^{n-2}02]$. As the n circular permutations of $[(20)^k 1]$ together have at least $n+1$ neighbors (they are not initial nor final), we see that P_0 must occur as one of the neighbors. Now applying the same argument to the circular permutations of $[(02)^k 1]$, and realizing that P_0 cannot occur as neighbor again, we see that the final distribution in the turn sequence is some circular permutation of $[(02)^k 1]$. In particular, no circular permutation of $[(20)^{k-1}021]$ or $[02(20)^{k-1}1]$ is initial or final. But all neighbors of these $2n$ distributions are circular permutations of either $[1^{n-3}012]$ or $[1^{n-4}0112]$. These are only $2n$ possible neighbors, contradiction. □

Problem 4. We consider sequences $a_0, a_1, a_2, \ldots$ of real numbers that satisfy

$$a_n = 4a_{n-1}(1 - a_{n-1})$$

for all positive integers n. How many such sequences satisfy $a_{2016} = a_0$?

Proposed by Merlijn Staps (Universiteit Utrecht).

Solution. There are 2^{2016} such sequences. If $a_0 < 0$ we have $a_1 = 4a_0(1-a_0) < a_0$ because $4(1-a_0) > 1$. It then follows that $a_{2016} < a_{2015} < \cdots < a_1 < a_0$, so we cannot have $a_{2016} = a_0$. If $a_0 > 1$ we have $a_1 = 4a_0(1-a_0) < 0$ and it follows that $a_{2016} < 0 < a_0$. Hence if $a_{2016} = a_0$ we must have $a_0 \in [0,1]$. This means that we can write $a_0 = \sin^2(\alpha)$ for some $\alpha \in [0, \frac{\pi}{2}]$. If $a_{n-1} = \sin^2(\beta)$ we have $a_n = 4a_{n-1}(1-a_{n-1}) = 4\sin^2(\beta)(1-\sin^2(\beta)) = 4\sin^2(\beta)\cos^2(\beta) = (2\sin(\beta)\cos(\beta))^2 = \sin^2(2\beta)$. By induction, it follows that $a_n = \sin^2(2^n\alpha)$ for all $n \geq 0$. In particular, we have $a_{2016} = \sin^2(2^{2016}\alpha)$. From $a_0 = a_{2016}$ it now follows that $2^{2016}\alpha = \pm\alpha + k\pi$ where k is an integer. This means that $\alpha = \pi \cdot \frac{k}{2^{2016}\pm 1}$. For $\alpha = \frac{k\pi}{2^{2016}-1}$ we must have $0 \leq k \leq \frac{2^{2016}-1}{2} = 2^{2015} - \frac{1}{2}$, which is satisfied for 2^{2015} values of k. For $\alpha = \frac{k\pi}{2^{2016}+1}$ we must have $0 \leq k \leq \frac{2^{2016}+1}{2} = 2^{2015} + \frac{1}{2}$, which is satisfied for $2^{2015}+1$ values of k. Because $2^{2016}+1$ and $2^{2016}-1$ are coprime only the value $\alpha = 0$ is counted twice, so in total there are $2^{2015} + (2^{2015}+1) - 1 = 2^{2016}$ possible values for α. This means that there are also 2^{2016} possible sequences. □

Problem 5. We are given N weights, with masses 1 kg, 2 kg, ..., N kg. We want to select at least two of these weights, such that their total mass equals the average mass of the other weights. Show that this is possible if and only if $N+1$ is a square.

Proposed by Arne Smeets (Katholieke Universiteit Leuven).

Solution. Suppose we can select weights such that the condition holds. Let $k \geq 2$ be the number of selected weights and let S be the sum of their masses. Then we must have $N \geq S \geq 1 + 2 + \ldots + k = \frac{k(k+1)}{2}$. Furthermore, we have $S = \frac{\frac{N(N+1)}{2} - S}{N-k}$, which rewrites to $2S(N-k+1) = N(N+1)$. It follows that $N-k+1$ divides $N(N+1)$, hence it also divides

$$N(N+1) - (N+k)(N-k+1) = k(k-1).$$

We have $k(k-1) \leq 2N - 2k < 2(N-k+1)$, so we must have $N-k+1 = k(k-1)$ and $N+1 = k^2$.
Conversely, if $N+1 = k^2$ then we can select the weights with masses 1 kg, ..., k kg. □

Problem 6. Decide whether there exists a function $f : \mathbb{R} \to \mathbb{Z}$ that is surjective on every infinite additive subgroup of $\mathbb{R}$.

Proposed by Merlijn Staps (Universiteit Utrecht).

Solution. Yes, such a function exists and can be constructed explicitly. For a rational number $q \notin \{0, \pm 1\}$ we define $p(q)$ as the largest prime number for which $e_p(q)$ (the number of factors p in q) is nonzero. Then we define

$$f(x) = \begin{cases} 0 & \text{if } x \in \{0, \pm 1\}; \\ \operatorname{sgn}(x) e_{p(x)}(x) & \text{if } x \in \mathbb{Q},\ x \notin \{0, \pm 1\}; \\ \operatorname{sgn}(x) \left\lfloor \frac{1}{\{|x|\}} \right\rfloor & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(Note that f is well-defined because the fractional part $\{x\}$ of x is never zero for irrational x.) It is now sufficient to show that f is surjective on every cyclic infinite additive subgroup of $\mathbb{R}$. Let $y > 0$ be a real number, we will show that f is surjective on $\langle y \rangle$. Note that f is an odd function. Therefore, it is sufficient to show that f takes all positive integer values on $\langle y \rangle$. First suppose that y is a rational number. Let q be a prime number that is larger than all prime factors of y. Then for a positive integer n we have $f(yq^n) = n$ since $yq^n \in \mathbb{Q}_{>0}$ and $p(yq^n) = q$. This means that f is surjective on $\langle y \rangle$ for rational y. Now suppose that $y > 0$ is irrational. Using the pigeonhole principle, it is straightforward to show that the set $\{\{my\} : m \in \mathbb{Z}_{>0}\}$ is dense in $[0, 1]$. Therefore there exists a positive integer m for which $\frac{1}{n+1} < \{my\} < \frac{1}{n}$. It then follows that $f(my) = \lfloor \frac{1}{\{my\}} \rfloor = n$, so f is also surjective on $\langle y \rangle$ for irrational y. $\square$